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TWO-SIDED CLIFFORD-VALUED SPECIAL AFFINE FOURIER TRANSFORM: PROPERTIES AND ASSOCIATED CONVOLUTION

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ABSTRACT. The primary focus of the present study is to analyse the Clifford-valued functions by introducing the notion of a two-sided Clifford-valued special affine Fourier transform in $L^2(\mathbb{R}^n, \mathcal{C}l_{0,n})$. Firstly we propose the novel definition of the Clifford-valued special affine Fourier transform and derive its fundamental properties which include inversion formula, translation covariance, scaling covariance and Plancherel formula. Subsequently we introduce the boundedness, continuity and differentiation theorems for the proposed transform. Finally we culminate our investigation by deriving the convolution for this newly proposed transform.

1. INTRODUCTION

In the theory of signal processing the Fourier transform has been widely used for the description of the input-output relationships of the linear filters [6] but unfortunately its applications got restricted due to at least two reasons: first it gives no information about the occurrence of the frequency component at a specific time; second it does not deal with the geometric features of the signals to be analysed. Due to this unavoidable fact some new transformations came into being, which include the fractional Fourier transform, chirp-based integral transform, linear canonical transform, LCT(affine Fourier transform) [5] and the six parameteric (a, b, c, d, p, q) special affine Fourier transform or offset Fourier transform which is a generalization of linear canonical transform. For a matrix parameter $A = \begin{bmatrix} a & b| & p \\ c & d| & q \end{bmatrix}$, the special affine Fourier transform (SAFT)of any signal h is defined as

$$\mathcal{O}_A[h(x)](w) = \int h(x)\mathcal{K}_A(x, w)dx \tag{1.1}$$

where $\mathcal{K}_A(x, w)$ denotes the kernel of the SAFT and is given by

$$\mathcal{K}_A(x, w) = \frac{1}{\sqrt{2\pi ib}} exp\left\{\frac{i}{2b}\left(ax^2 - 2x(w-p) - 2w(dp-bq) + d(w^2+p^2)\right)\right\} \tag{1.2}$$

with $ad - bc = 1$ and $b \neq 0$.
As a special case one can easily get linear canonical transform, LCT [7,8,22,23,26]. The application part of SAFT is similar to LCT but due to extra two parameters of SAFT, it is more general and flexible than LCT. Therefore, it has attained more popularity in signal and image processing [9–11].
It is well known that multi-variate signals are inherited by some qualitative geometric features,however,

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the quaternion algebra has a limitation to deal with such geometric features and is therefore not suitable for an excellent detection of the edges and corners which arise in multi-dimensional or higher dimensional signals. To address such limitations a new theory came into existence, that is the theory of Clifford algebra, which is not only characterized by advantages of both the Grassmann exterior algebra and Hamiltons algebra of quaternions but also inherits a function theory which is a higher dimensional analog of the theory of holomorphic functions of one complex variable.

The Clifford algebra is an associative non-commutative algebra having zero divisors that includes the algebraic properties of real numbers, complex numbers and quaternions along with the geometric properties of Grassman algebra [12, 24, 25]. From the last two decades, the Clifford algebra has attained a special status in the analysis of higher dimensional signals due to the fact that such algebras encompass all dimensions at once unlike to the multi-dimensional tensorial approach with tensor products of one-dimensional phenomena [13]. Such as Brackx et al in. [14] constructed a pair of Clifford-Fourier transforms based on the operator exponentials in the framework of Clifford analysis and Hitzer and Bahri [15] have constructed a generalized real Fourier transform for Clifford multivector- valued functions for $n = 2, 3(\bmod 4)$ over $\mathbb{R}^n$. They not only studied the basis properties of the proposed transform but also formulated several inequalities in the Clifford-Fourier domain. Later on it was De Bie and De Schepper [16] who have introduced Fractional Fourier transform based on two numerical parameters α and β and have obtained an explicit expression for the kernel in terms of Bessel functions. Recently, Haipan Shi et al. [17] derived uncertainty principles of the fractional Clifford-Fourier transform and Shie et al. [4] proposed a new version of fractional Clifford Fourier transform and derived its several properties, especially formulated a new differentiation theorem.In the recent times, yang et al. [18] have introduced the notion of the two-sided fractional Clifford-Fourier transform and constructed a relationship of it with the two-sided Clifford Fourier transform. Very recently Shah and Teali [18] studied the Clifford-valued linear canonical transform, convolution and its uncertainty principles and discussed the basic properties of two-sided Clifford-valued linear canonical transform and derived the mustard convolution.

Keeping in mind the more generality and more flexibility of the special affine transform and profound characteristics of Clifford algebra, we are deeply motivated to introduce a novel integral transform called as two-sided Clifford- valued special affine Fourier transform.

The rest of the sequence of the article is as follows: Section 2 deals briefly with preliminary aspects of the Clifford algebra, definition and basic properties of Clifford Fourier transform. Section 3 is concerned with the novel definition of two-sided Clifford-valued special affine Fourier transform and its fundamental properties. In section 4, boundedness, continuity and differentiation theorems for this novel transform have been discussed. The convolution associated with two-sided Clifford-valued special affine transform is also formulated in section 4. Finally, a conclusion is derived in Section 5.

2. Preliminaries

In this section, we shall remind the basic notions of Clifford algebra which are going to serve as a building block for the subsequent developments.

The Clifford algebra $\mathcal{C}l_{0,n}$ is an associative but non-commutative algebra generated by the orthonormal basis $\{f_1, f_2, ..., f_n\}$ of the n dimensional Euclidean space $\mathbb{R}^n$ executed by the multiplication rule:

$$f_i f_j + f_j f_i = -2\delta_{ij}, \qquad i, j = 1, 2, ..., n,$$

where δ_{ij} denotes the Kronecker's delta function. Generally speaking, every element of a Clifford algebra is called a multi-vector and a multi-vector $h \in \mathcal{C}l_{0,n}$ can be expressed in the following form:

$$h=\sum_{M} h_M f_M = \langle h\rangle_0 + \langle h\rangle_1 + ... + \langle h\rangle_n, \quad h_M \in \mathbb{R}, \quad M \subset \{1,2,...,n\},$$

where $f_M = f_{i_1} f_{i_2} ... f_{i_k}$ and $i_1 \leq i_2 \leq ... i_k$. Further $\langle . \rangle_k$ is called as the grade k- part of h. In particular $\langle . \rangle_0, \langle . \rangle_1, \langle . \rangle_2$,... respectively called as the scalar part, vector part, bivector part, and so on. Furthermore, we can define the reverse of $h \in \mathcal{C}l_{0,n}$ [19] as;

$$\tilde{h} = \sum_{m=0}^{n} (-1)^{\frac{m(m-1)}{2}} \langle h \rangle_m.$$

Therefore, the square norm of h is defined by

$$\|h\|^2 = \langle h\tilde{h} \rangle.$$

Also, the Clifford conjugate of $h \in \mathcal{C}l_{0,n}$ is defined in [1] by

$$\overline{h} = \sum_{m=0}^{n} (-1)^{\frac{m(m-1)}{2}} \overline{\langle h \rangle}_m.$$

Where $\overline{h}$ means to change in the basis decomposition of h the sign of every vector of negative square $\overline{f_M} = \epsilon_{\alpha_1} f_{\alpha_1} \epsilon_{\alpha_2} f_{\alpha_2} ... \epsilon_{\alpha_m} f_{\alpha_m}, \quad 1 \leq \alpha_1 \leq \alpha_2 \leq ... \leq \alpha_m \leq n$, [21].

We now recap the definition of two-sided Clifford- valued Fourier transform and two-sided Clifford-valued linear canonical transform and recaptulate their basic properties.

Definition 2.1. The two-sided Clifford-valued Fourier transform of any $h \in L^2(\mathbb{R}^n, \mathcal{C}l_{0,n})$ is denoted by $\mathcal{F}_c[h(\mathbf{x})]$ and is defined in [4] by

$$\mathcal{F}_c[h(\mathbf{x})](\mathbf{w}) = \int_{\mathbb{R}^n} \prod_{i=1}^{k} exp\{-f_i x_i w_i\} h(\mathbf{x}) \prod_{j=k+1}^{n} exp\{-f_j x_j w_j\} d^n\mathbf{x}, \quad \mathbf{x,w} \in \mathbb{R}^n.$$

Some important properties of the Clifford Fourier transform (2.1) are listed below

(1). For any $h_1, h_2 \in L^2(\mathbb{R}^n, \mathcal{C}l_{0,n})$, we have

$$\langle h_1, h_2 \rangle_{L^2(\mathbb{R}^n, \mathcal{C}l_{0,n})} = \Big\langle \mathcal{F}_c[h_1], \mathcal{F}_c[h_2] \Big\rangle_{L^2(\mathbb{R}^n, \mathcal{C}l_{0,n})}. \tag{2.1}$$

This is called Plancherelformula and for its proof, visit [20]

(2). Every Clifford-valued function $h \in L^2(\mathbb{R}^n, \mathcal{C}l_{o,n})$ can be reconstructed by the given formula

$$h(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \prod_{i=1}^{k} exp\{f_i x_i w_i\} \mathcal{F}_c[h(\mathbf{x})] \prod_{j=k+1}^{n} exp\{f_j x_j w_j\} d\mathbf{w}. \tag{2.2}$$

Definition 2.2. [1] Let $f, g \in \mathcal{C}l_{0,n}$ denote square roots of unity i.e.$f^2 = g^2 = -1$ and $A_k = \begin{bmatrix} a_k & b_k \\ c_k & d_k \end{bmatrix}$ be a matrix parameter with $a_k, b_k, c_k, d_k \in \mathbb{R}$ and $a_k d_k - b_k c_k = 1$ for $k = 1, 2, ..., n$, then the two-sided Clifford-valued linear canonical transform of any Clifford- valued function $h \in L^1(\mathbb{R}^n, \mathcal{C}l_{0,n})$ with respect to f and g is defined by

$$\mathcal{L}_{\mathbf{c}}[h(\mathbf{x})](\mathbf{w}) = \int_{\mathbb{R}^n} \prod_{i=1}^{k} \mathcal{K}_{A_i}^f(x_i, w_i) h(\mathbf{x}) \prod_{j=k+1}^{n} \mathcal{K}_{A_j}^g(x_j, w_j) d^n\mathbf{x}, \tag{2.3}$$

where **x,w** $\in \mathbb{R}^n$, $b_k \neq 0, 1 \leq k \leq n$ and the left and right kernels are respectively given by

$$\mathcal{K}^f_{A_i}(x_i, w_i) = \frac{1}{\sqrt{(2\pi)^k b_i}} exp\left\{\frac{f}{2b_i}\left(a_i x_i^2 - 2x_i w_i + d_i w_i^2 - \frac{k\pi b_i}{2}\right)\right\},$$

$$\mathcal{K}^g_{A_j}(x_i, w_i) = \frac{1}{\sqrt{(2\pi)^{n-k} b_j}} exp\left\{\frac{g}{2b_j}\left(a_j x_j^2 - 2x_j w_j + d_j w_j^2 - \frac{(n-k)\pi b_j}{2}\right)\right\}.$$

The basic properties of Clifford-valued linear canonical transform have been already discussed in [1] and [18].

Now we are all set to investigate our main work related to two-sided Clifford valued offset linear canonical transform or special affine Fourier transform.

3. Two-sided Clifford-valued Special Affine Fourier Transform

This section includes the concept of two-sided Clifford-valued special affine Fourier transform and its fundamental properties. Let us begin by the definition of two-sided Clifford-valued special affine Fourier (offset linear canonical) transform.

Definition 3.1. Let $f, g \in \mathcal{C}l_{0,n}$ denote square roots of unity i.e.$f^2 = g^2 = -1$ and $A_k = \begin{bmatrix} a_k & b_k| & p_k \\ c_k & d_k| & q_k \end{bmatrix}$ be a matrix parameter with $a_k, b_k, c_k, d_k, p_k, q_k \in \mathbb{R}$ and $a_k d_k - b_k c_k = 1$ for $k = 1, 2, ..., n$, then the two-sided Clifford-valued offset linear canonical transform of any Clifford- valued function $h \in L^1(\mathbb{R}^n, \mathcal{C}l_{0,n})$ with respect to f and g is defined by

$$\mathcal{O}_{\mathbf{c}}[h(\mathbf{x})](\mathbf{w}) = \int_{\mathbb{R}^n} \prod_{i=1}^{k} \mathcal{K}^f_{A_i}(x_i, w_i) h(\mathbf{x}) \prod_{j=k+1}^{n} \mathcal{K}^g_{A_j}(x_j, w_j) d^n\mathbf{x}, \tag{3.1}$$

where **x,w** $\in \mathbb{R}^n$, $b_k \neq 0, 1 \leq k \leq n$ and the left and right kernels are respectively given by

$$\mathcal{K}^f_{A_i}(x_i, w_i) = \frac{1}{\sqrt{(2\pi)^k b_i}} e^{\left\{\frac{f}{2b_i}\left(a_i x_i^2 - 2x_i(w_i - p_i) - 2w_i(d_i p_i - b_i q_i) + d_i(w_i^2 + p_i^2) - \frac{k\pi b_i}{2}\right)\right\}},$$

$$\mathcal{K}^g_{A_j}(x_i, w_i) = \frac{1}{\sqrt{(2\pi)^{n-k} b_j}} e^{\left\{\frac{g}{2b_j}\left(a_j x_j^2 - 2x_j(w_j - p_j) - 2w_j(d_j p_j - b_j q_j) + d_j(w_j^2 + p_j^2) - \frac{(n-k)\pi b_j}{2}\right)\right\}}.$$

It is worth to mention that the Definition 3.1 reduces to the following integral transforms: (i) Two- sided Clifford valued linear canonical transform when $A_k = \begin{bmatrix} a_k & b_k| & 0 \\ c_k & d_k| & 0 \end{bmatrix}, 1 \leq k \leq n$ see [1].

(ii) Two- sided fractional Clifford-valued Fourier transform when
$A_k = \begin{bmatrix} cos\theta_k & sin\theta_k| & 0 \\ -sin\theta_k & cos\theta_k| & 0 \end{bmatrix}, 1 \leq k \leq n$, see [2].

(iii) Two-sided Clifford-valued Fourier transform when $A_k = \begin{bmatrix} 0 & 1| & 0 \\ -1 & 0| & 0 \end{bmatrix}, 1 \leq k \leq n$, see [3].

(iv). If both the kernels are placed on the left or right side of the above Definition 3.1, then we get left or right sided Clifford-valued special affine Fourier transform.

The following theorem investigates the inverse transformation of the proposed two-sided Clifford-valued special affine Fourier transform (3.1).

Theorem 3.2. *Let $\mathcal{O}_{\mathbf{c}}[h(\boldsymbol{x})](\boldsymbol{w})$ be the two-sided Clifford-valued special affine Fourier transform of any Clifford-valued function $h \in L^2(\mathbb{R}^n, \mathcal{C}l_{0,n})$, then $h(\boldsymbol{x})$ can be reconstructed by the formula*

$$h(\boldsymbol{x}) = \int_{\mathbb{R}^n} \prod_{i=1}^{k} \mathcal{K}_{A_i}^{-f}(x_i, w_i)\mathcal{O}_{\mathcal{C}}[h(\boldsymbol{y})](\boldsymbol{w}) \prod_{j=k+1}^{n} \mathcal{K}_{A_j}^{-g}(x_j, w_j)d^n\boldsymbol{w}. \tag{3.2}$$

Proof. Using Definition 3.1, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} \prod_{i=1}^{k} \mathcal{K}_{A_i}^{-f}(x_i, w_i)\mathcal{O}_{\mathcal{C}}[h(\mathbf{y})](\mathbf{w}) \prod_{j=k+1}^{n} \mathcal{K}_{A_j}^{-g}(x_j, w_j)d^n\mathbf{w} \\
&= \int_{\mathbb{R}^n} \prod_{i=1}^{k} \mathcal{K}_{A_i}^{-f}(x_i, w_i) \int_{\mathbb{R}^n} \prod_{i=1}^{k} \mathcal{K}_{A_i}^{f}(y_i, w_i)h(\mathbf{y}) \prod_{j=k+1}^{n} \mathcal{K}_{A_j}^{g}(y_j, w_j)d^n\mathbf{y} \prod_{j=k+1}^{n} K_{A_j}^{-g}(x_j, w_j)d^n\mathbf{w} \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \prod_{i=1}^{k} \frac{1}{(2\pi)^k b_i} exp\left\{\frac{-f}{2b_i}\left(a_i(x_i^2 - y_i^2) - 2(x_i - y_i)(w_i - p_i)\right)\right\} h(\mathbf{y}) \\
&\times \prod_{j=k+1}^{n} \frac{1}{(2\pi)^{n-k} b_j} exp\left\{\frac{g}{2b_j}\left(a_j(y_j^2 - x_j^2) - 2(y_j - x_j)(w_j - p_j)\right)\right\} d^n\mathbf{y}d^n\mathbf{w} \\
&= \int_{\mathbb{R}^n} \prod_{i=1}^{k} \frac{1}{(2\pi)^k b_i} exp\left\{\frac{-f}{2b_i}\left(a_i(x_i^2 - y_i^2) + 2p_i(x_i - y_i)\right)\right\} \int_{\mathbb{R}^k} e^{f(x_i - y_i)\frac{w_i}{b_i}} d^k\mathbf{w}h(\mathbf{y}) \\
&\times \prod_{j=k+1}^{n} \frac{1}{(2\pi)^{n-k} b_j} exp\left\{\frac{g}{2b_j}\left(a_j(y_j^2 - x_j^2) + 2p_j(y_j - x_j)\right)\right\} \int_{\mathbb{R}^k} e^{g(x_j - y_j)\frac{w_j}{b_j}} d^{n-k}\mathbf{w}d^n\mathbf{y} \\
&= \int_{\mathbb{R}^n} \prod_{i=1}^{k} \frac{1}{(2\pi)^k b_i} exp\left\{\frac{-f}{2b_i}\left(a_i(x_i^2 - y_i^2) + 2p_i(x_i - y_i)\right)\right\} \prod_{i=1}^{k} (2\pi)^k b_i \delta(y_i - x_i)h(\mathbf{y}) \\
&\times \prod_{j=k+1}^{n} \frac{1}{(2\pi)^{n-k} b_j} exp\left\{\frac{g}{2b_j}\left(a_j(y_j^2 - x_j^2) + 2p_j(y_j - x_j)\right)\right\} \prod_{j=k+1}^{n} (2\pi)^{n-k} b_j \delta(y_j - x_j)d^n\mathbf{y} \\
&= \int_{\mathbb{R}^n} \prod_{i=1}^{k} exp\left\{\frac{-f}{2b_i}\left(a_i(x_i^2 - y_i^2) + 2p_i(x_i - y_i)\right)\right\} \prod_{i=1}^{k} \delta(y_i - x_i)h(\mathbf{y}) \\
&\times \prod_{j=k+1}^{n} exp\left\{\frac{g}{2b_j}\left(a_j(y_j^2 - x_j^2) + 2p_j(y_j - x_j)\right)\right\} \prod_{j=k+1}^{n} \delta(y_j - x_j)d^n\mathbf{y} \\
&= h(\mathbf{x}).
\end{aligned}$$

This completes the proof. □

Corollary:
The commutative relationship between the offset linear canonical transform $\mathcal{O}_{\mathcal{C}}$ and its inverse $\mathcal{O}_{\mathcal{C}}^{-1}$ is given by

$$h(\mathbf{x}) = \mathcal{O}_{\mathcal{C}}\left[\mathcal{O}_{\mathcal{C}}^{-1}\{h(\mathbf{y})\}(\mathbf{w})\right](\mathbf{x}) = \mathcal{O}_{\mathcal{C}}^{-1}\left[\mathcal{O}_{\mathcal{C}}\{h(\mathbf{y})\}(\mathbf{w})\right](\mathbf{x}). \tag{3.3}$$

Now we shall investigate some fundamental properties of the proposed two-sided Clifford-valued special affine Fourier transform.

Theorem 3.3. *(**Translation Covariance**) For any Clifford-valued function $h \in L^2(\mathbb{R}^n, \mathcal{C}l_{0,n})$ and $t \in \mathbb{R}^n$, the following relation holds*

$$\mathcal{O}_{\mathcal{C}}[h(\boldsymbol{x}\text{-}\boldsymbol{t})](\boldsymbol{w})$$
$$= \prod_{i=1}^{k} exp\left\{\frac{f}{2b_i}(a_i t_i^2 - 2t_i(w_i - p_i))\right\} \mathcal{O}_{\mathcal{C}}\left[\prod_{i=1}^{k} exp\left\{\frac{f a_i y_i t_i}{b_i}\right\} h(\boldsymbol{y}) \prod_{j=k+1}^{n} exp\left\{\frac{g a_j y_j t_j}{b_j}\right\}\right](\boldsymbol{w})$$
$$\times \prod_{j=k+1}^{n} exp\left\{\frac{g}{2b_j}(a_j t_j^2 - 2t_j(w_j - p_j))\right\}.$$

Proof. By using Definition 3.1, we have

$$\mathcal{O}_{\mathcal{C}}[h(\mathbf{x}\text{-}\mathbf{t})](\mathbf{w}) = \int_{\mathbb{R}^n} \prod_{i=1}^{k} \mathcal{K}_{A_i}^{f}(x_i, w_i) h(\mathbf{x}\text{-}\mathbf{t}) \prod_{j=k+1}^{n} \mathcal{K}_{A_j}^{g}(x_j, w_j) d^n\mathbf{x}. \tag{3.4}$$

Set $\mathbf{x}\text{-}\mathbf{t} = \mathbf{y}$, so that $d\mathbf{x} = d\mathbf{y}$, we have from equation (3.4)

$$\mathcal{O}_{\mathcal{C}}[h(\mathbf{x}\text{-}\mathbf{t})](\mathbf{w})$$
$$= \int_{\mathbb{R}^n} \prod_{i=1}^{k} \mathcal{K}_{A_i}^{f}(x_i, w_i) h(\mathbf{y}) \prod_{j=k+1}^{n} \mathcal{K}_{A_j}^{g}(x_j, w_j) d^n\mathbf{y}$$
$$= \int_{\mathbb{R}^n} \prod_{i=1}^{k} exp\left\{\frac{f}{2b_i}\left(a_i(y_i + t_i)^2 - 2(y_i + t_i)(w_i - p_i) - 2w_i(d_i p_i - b_i q_i) + d_i(w_i + p_i^2)\right) - \frac{k\pi b_i}{2}\right\}$$
$$\times \frac{1}{\sqrt{(2\pi)^k b_i}} h(\mathbf{y})$$
$$\times \prod_{j=k+1}^{n} \frac{1}{\sqrt{(2\pi)^{n-k} b_j}} exp\left\{\frac{g}{2b_j}\left(a_j(y_j + t_j)^2 - 2(y_j + t_j)(w_j - p_j) - 2w_j(d_j p_j - b_j q_j)\right.\right.$$
$$\left.\left. + d_j(w_j + p_j^2)\right) - \frac{(n-k)\pi b_j}{2}\right\} d^n\mathbf{y}$$
$$= \prod_{i=1}^{k} exp\left\{\frac{f}{2b_i}(a_i t_i^2 - 2t_i(w_i - p_i))\right\}$$
$$\times \frac{1}{\sqrt{(2\pi)^k b_i}} exp\left\{\frac{f}{2b_i}\left(a_i y_i^2 - 2y_i(w_i - p_i) - 2w_i(d_i p_i - b_i q_i) + d_i(w_i + p_i^2) - \frac{k\pi b_i}{2}\right)\right\}$$
$$\times \prod_{i=1}^{k} exp\left\{\frac{f a_i y_i t_i}{b_i}\right\} h(\mathbf{y}) \prod_{j=k+1}^{n} exp\left\{\frac{g a_j y_j t_j}{b_j}\right\}$$
$$\times \frac{1}{\sqrt{(2\pi)^{n-k} b_j}} exp\left\{\frac{g}{2b_j}\left(a_j y_j^2 - 2y_j(w_j - p_j) - 2w_j(d_j p_j - b_j q_j) + d_j(w_j + p_j^2) - \frac{(n-k)\pi b_j}{2}\right)\right\}$$
$$\times d^n\mathbf{y} \prod_{j=k+1}^{n} exp\left\{\frac{g}{2b_j}(a_j t_j^2 - 2t_j(w_j - p_j))\right\}$$
$$= \prod_{i=1}^{k} exp\left\{\frac{f}{2b_i}(a_i t_i^2 - 2t_i(w_i - p_i))\right\} \int_{\mathbb{R}^n} \prod_{i=1}^{k} \mathcal{K}_{A_i}^{f}(y_i, w_i) \prod_{i=1}^{k} exp\left\{\frac{f a_i y_i t_i}{b_i}\right\}$$
$$\times h(\mathbf{y}) \prod_{j=k+1}^{n} exp\left\{\frac{g a_j y_j t_j}{b_j}\right\} \prod_{j=k+1}^{n} \mathcal{K}_{A_i}^{g}(y_j, w_j)$$
$$\times \prod_{j=k+1}^{n} exp\left\{\frac{g}{2b_j}(a_j t_j^2 - 2t_j(w_j - p_j))\right\}$$
$$= \prod_{i=1}^{k} exp\left\{\frac{f}{2b_i}(a_i t_i^2 - 2t_i(w_i - p_i))\right\} \mathcal{O}_{\mathcal{C}}\left[\prod_{i=1}^{k} exp\left\{\frac{f a_i y_i t_i}{b_i}\right\} h(\mathbf{y}) \prod_{j=k+1}^{n} exp\left\{\frac{g a_j y_j t_j}{b_j}\right\}\right](\mathbf{w})$$
$$\times \prod_{j=k+1}^{n} exp\left\{\frac{g}{2b_j}(a_j t_j^2 - 2t_j(w_j - p_j))\right\}.$$

This proves the theorem. □

Theorem 3.4. *(**Scaling Covariance**) For any scaled function $h_\lambda(\boldsymbol{x}) = h(\lambda \boldsymbol{x}), \lambda \in \mathbb{R}^+$, the two-sided Clifford-valued offset linear canonical transform is given by*

$$\mathcal{O}_{\mathbf{c}}[h_\lambda(\boldsymbol{x})](\boldsymbol{w}) = \frac{1}{\lambda^n}\mathcal{O}_{\mathcal{C}}[h(\boldsymbol{y})]\left(\frac{\boldsymbol{w}}{\lambda}\right), \tag{3.5}$$

where the parameters used in RHS of the equation (3.5) are $A_{k'} = \begin{bmatrix} a_{k'} & b_k| & p_k \\ c_k & d_{k'}| & q_k \end{bmatrix}$ *with* $A_k^{'} = \frac{A_k}{\lambda}, d_k^{'} = \lambda^2 d_k$.

Proof. Inserting $h(\mathbf{x}) = h_\lambda(\mathbf{x})$ in Definition 3.1, we have

$$\begin{aligned}
&\mathcal{O}_{\mathbf{c}}[h_\lambda(\mathbf{x})](\mathbf{w}) \\
&= \int_{\mathbb{R}^n} \prod_{i=1}^{k} \mathcal{K}^f_{A_i}(x_i, w_i) h(\lambda \mathbf{x}) \prod_{j=k+1}^{n} \mathcal{K}^g_{A_j}(x_j, w_j) d^n\mathbf{x} \\
&= \int_{\mathbb{R}^n} \prod_{i=1}^{k} \mathcal{K}^f_{A_i}\left(\frac{y_i}{\lambda}, w_i\right) h(\mathbf{y}) \prod_{j=k+1}^{n} \mathcal{K}^g_{A_j}\left(\frac{y_j}{\lambda}, w_j\right) \frac{d^n\mathbf{y}}{\lambda^n} \\
&= \int_{\mathbb{R}^n} \prod_{i=1}^{k} \frac{1}{\sqrt{(2\pi)^k b_i}} exp\left\{\frac{f}{2b_i}\left(a_i \frac{y_i^2}{\lambda^2} - 2\frac{y_i}{\lambda}(w_i - p_i) - 2w_i(d_i p_i - b_i q_i) + d_i(w_i^2 + p_i^2) - \frac{k\pi b_i}{2}\right)\right\} h(\mathbf{y}) \\
&\times \int_{\mathbb{R}^n} \prod_{j=k+1}^{n} \frac{1}{\sqrt{(2\pi)^{n-k} b_j}} exp\left\{\frac{g}{2b_j}\left(a_j \frac{y_j^2}{\lambda^2} - 2\frac{y_j}{\lambda}(w_j - p_j) - 2w_j(d_j p_j - b_j q_j)\right.\right. \\
&\qquad\qquad \left.\left. + d_j(w_j^2 + p_j^2) - \frac{(n-k)\pi b_j}{2}\right)\right\} \frac{d^n\mathbf{y}}{\lambda^n}.
\end{aligned}$$

On setting $a_i^{'} = \frac{a_i}{\lambda^2}, d_i^{'} = \lambda^2 d_i, a_j^{'} = \frac{a_j}{\lambda^2}, d_j^{'} = \lambda^2 d_j$, we have

$$\begin{aligned}
&\mathcal{O}_{\mathbf{c}}[h_\lambda(\mathbf{x})](\mathbf{w}) \\
&= \int_{\mathbb{R}^n} \prod_{i=1}^{k} \frac{1}{\sqrt{(2\pi)^k b_i}} exp\left\{\frac{f}{2b_i}\left(a_i^{'} y_i^2 - 2y_i \frac{(w_i - p_i)}{\lambda} - 2\frac{w_i}{\lambda}\frac{(d_i^{'} p_i - b_i q_i)}{\lambda} + d_i^{'}\left(\frac{w_i^2}{\lambda^2} + \frac{p_i^2}{\lambda^2}\right) - \frac{k\pi b_i}{2}\right)\right\} \\
&\times h(\mathbf{y}) \int_{\mathbb{R}^n} \prod_{j=k+1}^{n} \frac{1}{\sqrt{(2\pi)^{n-k} b_j}} exp\left\{\frac{g}{2b_j}\left(a_j^{'} y_j^2 - 2y_j \frac{(w_j - p_j)}{\lambda} - 2\frac{w_j}{\lambda}\frac{(d_j^{'} p_j - b_j q_j)}{\lambda}\right.\right. \\
&\qquad\qquad \left.\left. + d_j^{'}\left(\frac{w_j^2}{\lambda^2} + \frac{p_j^2}{\lambda^2}\right) - \frac{(n-k)\pi b_j}{2}\right)\right\} \frac{d^n\mathbf{y}}{\lambda^n} \\
&= \int_{\mathbb{R}^n} \prod_{i=1}^{k} \mathcal{K}^f_{A_i^{'}}\left(y_i, \frac{w_i}{\lambda}\right) h(\mathbf{y}) \int_{\mathbb{R}^n} \prod_{j=k+1}^{n} \mathcal{K}^f_{A_j^{'}}\left(y_j, \frac{w_j}{\lambda}\right) \frac{d^n\mathbf{y}}{\lambda^n} \\
&= \frac{1}{\lambda^n}\mathcal{O}_{\mathcal{C}}[h(\mathbf{y})]\left(\frac{\mathbf{w}}{\lambda}\right)
\end{aligned}$$

where $A_{k'} = \begin{bmatrix} a_{k'} & b_k| & p_k \\ c_k & d_{k'}| & q_k \end{bmatrix}$ with $A_k^{'} = \frac{A_k}{\lambda}, d_k^{'} = \lambda^2 d_k$.
This ends the proof. □

Corollary (Reflection):
The two-sided Clifford-valued special affine Fourier transform of any function $h(-\mathbf{x})$ is given by

$$\mathcal{O}_{\mathbf{c}}[h(-\mathbf{x})](\mathbf{w}) = (-1)^n \mathcal{O}_{\mathcal{C}}[h(\mathbf{x})](\text{-}\mathbf{w}). \tag{3.6}$$

We shall now derive the Plancherel theorem for the proposed two-sided Clifford-valued offset linear canonical transform or special affine Fourier transform (3.1) under the condition $fh_1\overline{h_2} = h_1\overline{h_2}f$. Firstly we shall state a lemma.

Lemma 3.5. *[1] (Plancherel theorem for two-sided Clifford-valued linear canonical tranasform). Let $\mathcal{L}_c[h_1(\boldsymbol{x})](\boldsymbol{w})$ and $\mathcal{L}_c[h_1(\boldsymbol{x})](\boldsymbol{w})$ be the two-sided Clifford LCTs of $h_1(\boldsymbol{x})$ and $h_2(\boldsymbol{x}) \in L^2(\mathbb{R}^n, \mathcal{C}l_{0,n})$ respectively. Then we have*

$$\left\langle L_c[h_1(\boldsymbol{x})](\boldsymbol{w}), L_c[h_2(\boldsymbol{x})](\boldsymbol{w})\right\rangle_{L^2(\mathbb{R}^n,\mathcal{C}l_{0,n})} = \left\langle h_1, h_2\right\rangle_{L^2(\mathbb{R}^n,\mathcal{C}l_{0,n})} \tag{3.7}$$

whenever $fh_1\overline{h_2} = h_1\overline{h_2}f$

Theorem 3.6. *(Plancherel theorem) Let $h_1(\boldsymbol{x})$ and $h_2(\boldsymbol{x}) \in L^2(\mathbb{R}^n, \mathcal{C}l_{0,n})$ and their respective two-sided Clifford offset LCT's are $\mathcal{O}_c[h_1(\boldsymbol{x})](\boldsymbol{w})$ and $\mathcal{O}_c[h_1(\boldsymbol{x})](\boldsymbol{w})$, then we have*

$$\left\langle O_c[h_1(\boldsymbol{x})](\boldsymbol{w}), O_c[h_1(\boldsymbol{x})](\boldsymbol{w})\right\rangle_{L^2(\mathbb{R}^n,\mathcal{C}l_{0,n})} = \left\langle h_1, h_2\right\rangle_{L^2(\mathbb{R}^n,\mathcal{C}l_{0,n})}$$

whenever $fh_1\overline{h_2} = h_1\overline{h_2}f$ holds.

Proof. Applying the Definition 3.1, we have

$$\begin{aligned}
&\left\langle \mathcal{O}_c[h_1(\mathbf{x})](\mathbf{w}), \mathcal{O}_c[h_1(\mathbf{x})](\mathbf{w})\right\rangle_{L^2(\mathbb{R}^n,\mathcal{C}l_{0,n})}\\
&= \int_{\mathbb{R}^n} \mathcal{O}_c[h_1(\mathbf{x})](\mathbf{w})\overline{\mathcal{O}_c[h_2(\mathbf{y})](\mathbf{w})}d\mathbf{w}\\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \prod\nolimits_{i=1}^{k} \mathcal{K}^f_{A_i}(x_i,w_i)h_1(\mathbf{x}) \prod_{j=k+1}^{n} \mathcal{K}^g_{A_j}(x_j,w_j)d^n\mathbf{x}\\
&\overline{\times\int_{\mathbb{R}^n} \prod\nolimits_{i=1}^{k} \mathcal{K}^f_{A_i}(y_i,w_i)h_2(\mathbf{y}) \prod_{j=k+1}^{n} \mathcal{K}^g_{A_j}(y_j,w_j)d^n\mathbf{y}}d\mathbf{w}\\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \prod\nolimits_{i=1}^{k} \mathcal{K}^f_{A_i}(x_i,w_i)h_1(\mathbf{x}) \prod_{j=k+1}^{n} \mathcal{K}^g_{A_j}(x_j,w_j)\\
&\times \overline{\prod_{j=k+1}^{n} K^g_{A_j}(y_j,w_j)}\,\overline{h_2(\mathbf{y})}\,\overline{\prod_{i=1}^{k} K^f_{A_i}(y_i,w_i)}d^n\mathbf{x}d^n\mathbf{y}d\mathbf{w}\\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \prod\nolimits_{i=1}^{k} \mathcal{K}^f_{A_i}(x_i,w_i)h_1(\mathbf{x}) \prod_{j=k+1}^{n} \mathcal{K}^g_{A_j}(x_j,w_j)\\
&\times \prod_{j=k+1}^{n} K^{-g}_{A_j}(y_j,w_j)\overline{h_2(\mathbf{y})}\prod_{i=1}^{k} K^{-f}_{A_i}(y_i,w_i)d^n\mathbf{x}d^n\mathbf{y}d\mathbf{w}\\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \prod\nolimits_{i=1}^{k} \mathcal{K}^f_{A_i}(x_i,w_i)h_1(\mathbf{x}) \prod_{j=k+1}^{n} \frac{1}{(2\pi)^{n-k}b_j}\\
&\times exp\left\{\frac{g}{2b_j}\left(a_j(x_j^2-y_j^2)-2(x_j-y_j)(w_j-p_j)\right)\right\}\overline{h_2(\mathbf{y})}\prod_{i=1}^{k} K^{-f}_{A_i}(y_i,w_i)d^n\mathbf{x}d^n\mathbf{y}d\mathbf{w}\\
&= \int_{\mathbb{R}^k}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \prod\nolimits_{i=1}^{k} \mathcal{K}^f_{A_i}(x_i,w_i)h_1(\mathbf{x}) \prod_{j=k+1}^{n} \frac{1}{(2\pi)^{n-k}b_j}exp\left\{\frac{ga_j}{2b_j}(x_j^2-y_j^2)\right\}\\
&\times \prod_{j=k+1}^{n}\int_{\mathbb{R}^{n-k}} exp\left\{\frac{-g}{b_j}(x_j-y_j)(w_j-p_j)\right\}d^{n-k}\mathbf{w}\overline{h_2(\mathbf{y})}\prod_{i=1}^{k} K^{-f}_{A_i}(y_i,w_i)d^n\mathbf{x}d^n\mathbf{y}d^k\mathbf{w}\\
&= \int_{\mathbb{R}^k}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \prod\nolimits_{i=1}^{k} \mathcal{K}^f_{A_i}(x_i,w_i)h_1(\mathbf{x}) \prod_{j=k+1}^{n} \frac{1}{(2\pi)^{n-k}b_j}exp\left\{\frac{ga_j}{2b_j}(x_j^2-y_j^2)\right\}
\end{aligned}$$

$$\times \prod_{j=k+1}^{n}\left((2\pi)^{n-k}b_j\right)\delta(x_j-y_j)exp\left\{\frac{g}{b_j}(x_j-y_j)p_j\right\}\overline{h_2(\mathbf{y})}\prod_{i=1}^{k}K_{A_i}^{-f}(y_i,w_i)d^n\mathbf{x}d^n\mathbf{y}d^k\mathbf{w}$$

$$=\int_{\mathbb{R}^k}\int_{\mathbb{R}^k}\int_{\mathbb{R}^n}\prod_{i=1}^{k}\mathcal{K}_{A_i}^{f}(x_i,w_i)h_1(\mathbf{x})exp\left\{\frac{g}{b_j}(x_j-y_j)p_j\right\}\overline{h_2(y_1,y_2,...,y_k,x_{k+1},...,x_n)}$$

$$\times\prod_{i=1}^{k}K_{A_i}^{-f}(y_i,w_i)d^n\mathbf{x}d^n\mathbf{y}d^k\mathbf{w}$$

For $fh_1\overline{h_2}=h_1\overline{h_2}f$, the equation extends to

$$\left\langle\mathcal{O}_c[h_1(\mathbf{x})](\mathbf{w}),\mathcal{O}_c[h_1(\mathbf{x})](\mathbf{w})\right\rangle_{L^2(\mathbb{R}^n,\mathcal{C}l_{0,n})}$$

$$=\int_{\mathbb{R}^k}\int_{\mathbb{R}^k}\int_{\mathbb{R}^n}h_1(\mathbf{x})\overline{h_2(y_1,y_2,...,y_k,x_{k+1},...,x_n)}exp\left\{\frac{g}{b_j}(x_j-y_j)p_j\right\}$$

$$\times\prod_{i=1}^{k}K_{A_i}^{f}(x_i,w_i)K_{A_i}^{-f}(y_i,w_i)d^n\mathbf{x}d^n\mathbf{y}d^k\mathbf{w}$$

$$=\int_{\mathbb{R}^k}\int_{\mathbb{R}^n}h_1(\mathbf{x})\overline{h_2(y_1,y_2,...,y_k,x_{k+1},...,x_n)}exp\left\{\frac{g}{b_j}(x_j-y_j)p_j\right\}\prod_{i=1}^{k}\int_{\mathbb{R}^k}\frac{1}{(2\pi)^k b_i}$$

$$\times exp\left\{\frac{f}{2b_i}\left(a_i(x_i^2-y_i^2)-2(x_i-y_i)(w_i-p_i)\right)\right\}d^k\mathbf{w}d^n\mathbf{x}d^k\mathbf{y}$$

$$=\int_{\mathbb{R}^k}\int_{\mathbb{R}^n}h_1(\mathbf{x})\overline{h_2(y_1,y_2,...,y_k,x_{k+1},...,x_n)}exp\left\{\frac{g}{b_j}(x_j-y_j)p_j\right\}\prod_{i=1}^{k}\frac{1}{(2\pi)^k b_i}$$

$$\times exp\left\{\frac{fa_i}{2b_i}(x_i^2-y_i^2)\right\}\int_{\mathbb{R}^k}exp\left\{\frac{f}{2b_i}(y_i-x_i)(w_i-p_i)\right\}d^k\mathbf{w}d^n\mathbf{x}d^k\mathbf{y}$$

$$=\int_{\mathbb{R}^k}\int_{\mathbb{R}^n}h_1(\mathbf{x})\overline{h_2(y_1,y_2,...,y_k,x_{k+1},...,x_n)}exp\left\{\frac{g}{b_j}(x_j-y_j)p_j\right\}\prod_{i=1}^{k}\frac{1}{(2\pi)^k b_i}$$

$$\times exp\left\{\frac{fa_i}{2b_i}(x_i^2-y_i^2)\right\}exp\left\{\frac{f}{b_i}(x_i-y_i)p_i\right\}(2\pi)^k b_i\delta(x_i-y_i)d^n\mathbf{x}d^k\mathbf{y}$$

$$=\int_{\mathbb{R}^n}h_1(\mathbf{x})\overline{h_2(x_1,x_2,...,x_k,x_{k+1},...,x_n)}d^n\mathbf{x}$$

$$=\left\langle h_1,h_2\right\rangle_{L^2(\mathbb{R}^n,\mathcal{C}l_{0,n})}.$$

This ends the proof. □

Corollary (Parseval Formula): If in the above theorem $h_1=h_2=h\in L^2(\mathbb{R}^n,\mathcal{C}l_{0,n})$, we get the following formula known as Parseval formula

$$\|\mathcal{O}_c[h]\|^2_{L^2(\mathbb{R}^n,\mathcal{C}l_{0,n})}=\|h\|^2_{L^2(\mathbb{R}^n,\mathcal{C}l_{0,n})}.$$

4. Differentiation Theorems and Convolution for Two-sided Clifford-valued Special Affine Fourier Transform

This section begins with the boundedness and continuity of two-sided Clifford-valued special affine Fourier transform. In order to prove the boundedness, we shall two lemmas which have been already proved.

Lemma 4.1. *[21] Let $\lambda\in\mathcal{C}l(p,q)$ and $\mu\in\mathcal{C}l(p,q)$ with $p+q=n$ be a square root of unity, then*

$$|\lambda e^{-\mu u(x,y)}|\leq 2^n|\lambda|\left(1+|\mu|^2\right)^{\frac{1}{2}} \tag{4.1}$$

Lemma 4.2. *[18]. The Clifford-valued linear canonical transform of $f\in L^2(\mathbb{R}^n,\mathcal{C}l_{0,n})$ is bounded and continuous on $\mathbb{R}^n$*

Theorem 4.3. *The two-sided Clifford-valued offset linear canonical transform of $h \in L^2(\mathbb{R}^n, \mathcal{C}l_{0,n})$ is bounded and continuous on $\mathbb{R}^n$.*

Proof. Using the Definition 3.1, we have

$$
\begin{aligned}
&\Big|\mathcal{O}_c[h(\mathbf{x})](\mathbf{w})\Big| \\
&= \Big|\int_{\mathbb{R}^n}\prod_{i=1}^{k}\mathcal{K}^f_{A_i}(x_i,w_i)h(\mathbf{x})\prod_{j=k+1}^{n}\mathcal{K}^g_{A_j}(x_j,w_j)d^n\mathbf{x}\Big| \\
&= \Big|\int_{\mathbb{R}^n}\prod_{i=1}^{k}\frac{1}{\sqrt{(2\pi)^k b_i}}exp\left\{\frac{f_i}{2b_i}\left(a_ix_i^2-2x_i(w_i-p_i)-2w_i(d_ip_i-b_iq_i)+d_i(w_i^2+p_i^2)-\frac{k\pi b_i}{2}\right)\right\}h(\mathbf{x}) \\
&\times\prod_{j=k+1}^{n}\frac{1}{\sqrt{(2\pi)^{n-k}b_j}}exp\left\{\frac{f_j}{2b_j}\left(a_jx_j^2-2x_j(w_j-p_j)-2w_j(d_jp_j-b_jq_j)+d_j(w_j^2+p_j^2)\right.\right. \\
&\left.\left.-\frac{(n-k)\pi b_j}{2}\right)\right\}d^n\mathbf{x}\Big| \\
&\le 2^{k+n-k}\int_{\mathbb{R}^n}|2\pi|^{\frac{-k}{2}}|b_1b_2...b_k|^{\frac{-1}{2}} \\
&\times\Big|exp\left\{\frac{f_1}{2b_1}\left(a_1x_1^2-2x_1(w_1-p_1)-2w_1(d_1p_1-b_1q_1)+d_1(w_1^2+p_1^2)-\frac{k\pi b_1}{2}\right\}\right.\Big| \\
&\times\Big|exp\left\{\frac{f_2}{2b_2}\left(a_2x_2^2-2x_2(w_2-p_2)-2w_2(d_2p_2-b_2q_2)+d_2(w_2^2+p_2^2)-\frac{k\pi b_2}{2}\right\}\right.\Big| \\
&\times...\times\Big|exp\left\{\frac{f_k}{2b_k}\left(a_kx_k^2-2x_k(w_k-p_k)-2w_k(d_kp_k-b_kq_k)+d_k(w_k^2+p_k^2)-\frac{k\pi b_k}{2}\right\}\right.\Big||h(\mathbf{x})| \\
&\times|2\pi|^{\frac{-n+k}{2}}|b_{k+1}b_{k+2}...b_n|^{\frac{-1}{2}} \\
&\times\Big|exp\left\{\frac{f_{k+1}}{2b_{k+1}}\left(a_{k+1}x_{k+1}^2-2x_{k+1}(w_{k+1}-p_{k+1})-2w_{k+1}(d_{k+1}p_{k+1}-b_{k+1}q_{k+1})\right.\right. \\
&\left.\left.+d_{k+1}(w_{k+1}^2+p_{k+1}^2)-\frac{(n-k)\pi b_{k+1}}{2}\right\}\right.\Big| \\
&\times\Big|exp\left\{\frac{f_{k+2}}{2b_{k+2}}\left(a_{k+2}x_{k+2}^2-2x_{k+2}(w_{k+2}-p_{k+2})-2w_{k+2}(d_{k+2}p_{k+2}-b_{k+2}q_{k+2})\right.\right. \\
&\left.\left.+d_{k+2}(w_{k+2}^2+p_{k+2}^2)-\frac{(n-k)\pi b_{k+2}}{2}\right\}\right.\Big| \\
&\times...\times\Big|exp\left\{\frac{f_n}{2b_n}\left(a_nx_n^2-2x_n(w_n-p_n)-2w_n(d_np_n-b_nq_n)+d_n(w_n^2+p_n^2)-\frac{(n-k)\pi b_n}{2}\right\}\right.\Big|d^n\mathbf{x} \\
&\le 2^n\int_{\mathbb{R}^n}|2\pi|^{\frac{-k}{2}}|b_1b_2...b_k|^{\frac{-1}{2}} \\
&\times(1+|e_1|^2)^{\frac{1}{2}}(1+|e_2|^2)^{\frac{1}{2}}...(1+|e_k|^2)^{\frac{1}{2}}|h(\mathbf{x})| \\
&\times|2\pi|^{\frac{-n+k}{2}}(1+|e_{k+1}|^2)^{\frac{1}{2}}(1+|e_{k+2}|^2)^{\frac{1}{2}}...(1+|e_n|^2)^{\frac{1}{2}}d^n\mathbf{x} \\
&= \int_{\mathbb{R}^n}\prod_{m=1}^{n}\left|\frac{2}{\pi}\right|^{\frac{n}{2}}|b_m|^{\frac{-1}{2}}\left(1+|e_m|^2\right)^{\frac{1}{2}}|h(\mathbf{x})|d^n\mathbf{x}<\infty
\end{aligned}
$$

Therefore, $\mathcal{O}_c[h(\mathbf{x})](\mathbf{w})$ is bounded.

Now we shall prove the continuity of $\mathcal{O}_c[h(\mathbf{x})](\mathbf{w})$,

$$
\begin{aligned}
&\Big|\mathcal{O}_c[h(\mathbf{x})](\mathbf{u+w})-\mathcal{O}_c[h(\mathbf{x})](\mathbf{w})\Big| \\
&= \Big|\int_{\mathbb{R}^n}\prod_{i=1}^{k}\frac{1}{\sqrt{(2\pi)^k b_i}}exp\left\{\frac{f_i}{2b_i}\left(a_ix_i^2-2x_i(u_i+w_i-p_i)-2(u_i+w_i)(d_ip_i-b_iq_i)\right.\right. \\
&\left.\left.+d_i[(u_i+w_i)^2+p_i^2]-\frac{k\pi b_i}{2}\right)\right\}h(\mathbf{x})
\end{aligned}
$$

$$\times \prod_{j=k+1}^{n} \frac{1}{\sqrt{(2\pi)^{n-k} b_j}} exp\left\{ \frac{f_j}{2b_j} \left(a_j x_j^2 - 2x_j(u_j + w_j - p_j) - 2(u_j + w_j)(d_j p_j - b_j q_j) \right.\right.$$

$$\left.\left. + d_j[(u_j + w_j)^2 + p_j^2] - \frac{(n-k)\pi b_j}{2} \right) \right\} d^n \mathbf{x}$$

$$- \int_{\mathbb{R}^n} \prod_{i=1}^{k} \frac{1}{\sqrt{(2\pi)^k b_i}} exp\left\{ \frac{f_i}{2b_i} \left(a_i x_i^2 - 2x_i(w_i - p_i) - 2w_i(d_i p_i - b_i q_i) + d_i(w_i^2 + p_i^2) - \frac{k\pi b_i}{2} \right) \right\} h(\mathbf{x})$$

$$\times \prod_{j=k+1}^{n} \frac{1}{\sqrt{(2\pi)^{n-k} b_j}} exp\left\{ \frac{f_j}{2b_j} \left(a_j x_j^2 - 2x_j(w_j - p_j) - 2w_j(d_j p_j - b_j q_j) + d_j(w_j^2 + p_j^2) \right.\right.$$

$$\left.\left. - \frac{(n-k)\pi b_j}{2} \right) \right\} d^n \mathbf{x} \Bigg|$$

$$= \Bigg| \int_{\mathbb{R}^n} \prod_{i=1}^{k} \frac{1}{\sqrt{(2\pi)^k b_i}} exp\left\{ \frac{f_i}{2b_i} \left(a_i x_i^2 - 2x_i(w_i - p_i) - 2w_i(d_i p_i - b_i q_i) + d_i(w_i^2 + p_i^2) - \frac{k\pi b_i}{2} \right) \right\} h(\mathbf{x})$$

$$\times \prod_{j=k+1}^{n} \frac{1}{\sqrt{(2\pi)^{n-k} b_j}} exp\left\{ \frac{f_j}{2b_j} \left(a_j x_j^2 - 2x_j(w_j - p_j) - 2w_j(d_j p_j - b_j q_j) + d_j(w_j^2 + p_j^2) \right.\right.$$

$$\left.\left. - \frac{(n-k)\pi b_j}{2} \right) \right\}$$

$$\times \left[exp\left\{ \frac{f_i}{2b_i} \left(-2x_i u_i - 2u_i(d_i p_i - b_i q_i) + d_i(u_i^2 + 2u_i w_i + p_i^2) \right) \right\} \right.$$

$$\left. \times exp\left\{ \frac{f_j}{2b_j} \left(-2x_j u_j - 2u_j(d_j p_j - b_j q_j) + d_j(u_j^2 + 2u_j w_j + p_j^2) \right) \right\} - 1 \right] d^n \mathbf{x} \Bigg|$$

$$\le \int_{\mathbb{R}^n} \Bigg| \prod_{i=1}^{k} \frac{1}{\sqrt{(2\pi)^k b_i}} exp\left\{ \frac{f_i}{2b_i} \left(a_i x_i^2 - 2x_i(w_i - p_i) - 2w_i(d_i p_i - b_i q_i) + d_i(w_i^2 + p_i^2) - \frac{k\pi b_i}{2} \right) \right\} h(\mathbf{x})$$

$$\times \prod_{j=k+1}^{n} \frac{1}{\sqrt{(2\pi)^{n-k} b_j}} exp\left\{ \frac{f_j}{2b_j} \left(a_j x_j^2 - 2x_j(w_j - p_j) - 2w_j(d_j p_j - b_j q_j) + d_j(w_j^2 + p_j^2) \right.\right.$$

$$\left.\left. - \frac{(n-k)\pi b_j}{2} \right) \right\} \Bigg|$$

$$\times \Bigg| \left[exp\left\{ \frac{f_i}{2b_i} \left(-2x_i u_i - 2u_i(d_i p_i - b_i q_i) + d_i(u_i^2 + 2u_i w_i + p_i^2) \right) \right\} \right.$$

$$\left. \times exp\left\{ \frac{f_j}{2b_j} \left(-2x_j u_j - 2u_j(d_j p_j - b_j q_j) + d_j(u_j^2 + 2u_j w_j + p_j^2) \right) \right\} - 1 \right] \Bigg| d^n \mathbf{x}$$

$$\le \int_{\mathbb{R}^n} \prod_{i=1}^{k} \frac{2^n}{\sqrt{(2\pi)^k |b_i|}} \left(1 + |f_i|^2 \right)^{\frac{-k}{2}} h(\mathbf{x}) \prod_{j=k+1}^{n} \frac{1}{\sqrt{(2\pi)^{n-k} |b_j|}} \left(1 + |f_j|^2 \right)^{\frac{-(n-k)}{2}}$$

$$\times \Bigg| \left[exp\left\{ \frac{f_i}{2b_i} \left(-2x_i u_i - 2u_i(d_i p_i - b_i q_i) + d_i(u_i^2 + 2u_i w_i + p_i^2) \right) \right\} \right.$$

$$\left. \times exp\left\{ \frac{f_j}{2b_j} \left(-2x_j u_j - 2u_j(d_j p_j - b_j q_j) + d_j(u_j^2 + 2u_j w_j + p_j^2) \right) \right\} - 1 \right] \Bigg| d^n \mathbf{x}$$

By Applying the Lebesgue dominated convergence theorem, we have

$$\lim_{u \to 0} \Big| \mathcal{O}_c[h(\mathbf{x})](\mathbf{u+w}) - \mathcal{O}_c[h(\mathbf{x})](\mathbf{w}) \Big| = 0 \tag{4.2}$$

which proves that $\mathcal{O}_c[h(\mathbf{x})](\mathbf{w})$ is continuous on $\mathbb{R}^n$, which in turn completes the proof of the theorem. □

Before proving the differentiation theorems for the proposed transform, we shall define the notion of Schwartz space in $L^2(\mathbb{R}^n, \mathcal{C}l_{0,n})$ and state a lemma proved in [1].

Definition 4.4. Let $f \in \mathbb{C}^{\infty}(\mathbb{R}^n, \mathcal{C}l_{0,n})$- the set of all smooth functions, then the Schwartz space in $L^2(\mathbb{R}^n, \mathcal{C}l_{0,n})$ is defined as

$$\mathcal{S}(\mathbb{R}^n, \mathcal{C}l_{0,n}) = \left\{ f \in \mathbb{C}^{\infty}(\mathbb{R}^n, \mathcal{C}l_{0,n}); \sup_{\mathbf{x}\in\mathbb{R}^n} \left(1+|x|^k\right) \left| \frac{\partial^{\sum \alpha_i}[h(\mathbf{x})]}{\partial_{x_1}^{\alpha_1}...\partial_{x_n}^{\alpha_n}} \right| < \infty \right\},$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{R}^+ \times \mathbb{R}^{\cdot}.. \times \mathbb{R}^+$ and $k \in \mathbb{Z}^+$.

Lemma 4.5. *[1] Let $h \in \mathcal{S}(\mathbb{R}^n, \mathcal{C}l_{0,n})$, then for each $x_s, 1 \le s \le n, n \in \mathbb{N}$, the two-sided Clifford-valued linear canonical transform (2.2) satisfies the following relations*

$$(i) \quad \mathcal{L}_c\left[\frac{\partial h(\boldsymbol{x})}{\partial x_s}\right](w) = \frac{f}{b_s}\mathcal{L}_c\left[w_s h(\boldsymbol{x})\right](\boldsymbol{w}) - \frac{f a_s}{b_s}\mathcal{L}_c\left[x_s h(\boldsymbol{x})\right](\boldsymbol{w}), \quad 1 \le s \le k$$

$$(ii) \quad \mathcal{L}_c\left[\frac{\partial h(\boldsymbol{x})}{\partial x_s}\right](w) = \mathcal{L}_c\left[w_s h(\boldsymbol{x})\right](\boldsymbol{w})\frac{g}{b_s} - \mathcal{L}_c\left[x_s h(\boldsymbol{x})\right](\boldsymbol{w})\frac{g a_s}{b_s}, \quad k+1 \le s \le n.$$

Theorem 4.6. *Let $u \in \mathcal{S}(\mathbb{R}^n, \mathcal{C}l_{0,n})$, then for each $x_s, 1 \le s \le n, n \in \mathbb{N}$, the following relations are satisfied by the two-sided Clifford-valued special affine Fourier transform (3.1)*

$$(i) \quad \mathcal{O}_c\left[\frac{\partial u(\boldsymbol{x})}{\partial x_s}\right](w) = \frac{f}{b_s}\mathcal{O}_c\left[(w_s - p_s)u(\boldsymbol{x})\right](\boldsymbol{w}) - \frac{f a_s}{b_s}\mathcal{O}_c\left[x_s u(\boldsymbol{x})\right](\boldsymbol{w}), \quad 1 \le s \le k$$

$$(ii) \quad \mathcal{O}_c\left[\frac{\partial u(\boldsymbol{x})}{\partial x_s}\right](w) = \mathcal{O}_c\left[(w_s - p_s)u(\boldsymbol{x})\right](\boldsymbol{w})\frac{g}{b_s} - \mathcal{O}_c\left[x_s u(\boldsymbol{x})\right](\boldsymbol{w})\frac{g a_s}{b_s}, \quad k+1 \le s \le n.$$

Proof. We have for $1 \le s \le k$

$$\begin{aligned}
&\mathcal{O}_c\left[\frac{\partial u(\mathbf{x})}{\partial x_k}\right](w) \\
&= \int_{\mathbb{R}^n} \prod_{i=1}^{k} \mathcal{K}^f_{A_i}(x_i, w_i) \frac{\partial u(\mathbf{x})}{\partial x_k} \prod_{j=k+1}^{n} \mathcal{K}^g_{A_j}(x_j, w_j) d^n\mathbf{x} \\
&= \int_{\mathbb{R}^n} \prod_{i=1}^{k} \frac{1}{\sqrt{(2\pi)^k b_i}} exp\left\{\frac{f}{2b_i}\left(a_i x_i^2 - 2x_i(w_i - p_i) - 2w_i(d_i p_i - b_i q_i) + d_i(w_i^2 + p_i^2) - \frac{k\pi b_i}{2}\right)\right\} \\
&\times \frac{\partial u(\mathbf{x})}{\partial x_k} \prod_{j=k+1}^{n} \mathcal{K}^g_{A_j}(x_j, w_j) d^n\mathbf{x} \\
&= \int_{\mathbb{R}^{n-1}} \left[\int_{\mathbb{R}} \prod_{i=1}^{k} \frac{1}{\sqrt{(2\pi)^k b_i}} exp\left\{\frac{f}{2b_i}\left(a_i x_i^2 - 2x_i(w_i - p_i) - 2w_i(d_i p_i - b_i q_i)\right.\right.\right. \\
&\left.\left.\left. + d_i(w_i^2 + p_i^2) - \frac{k\pi b_i}{2}\right)\right\}\right] \prod_{j=k+1}^{n} \mathcal{K}^g_{A_j}(x_j, w_j) dx_1 dx_2 ... dx_s dx_{s+1} ... dx_n \\
&= \int_{\mathbb{R}^{n-1}} \left[\prod_{i=1}^{k} \frac{1}{\sqrt{(2\pi)^k b_i}} exp\left\{\frac{f}{2b_i}\left(a_i x_i^2 - 2x_i(w_i - p_i) - 2w_i(d_i p_i - b_i q_i) + d_i(w_i^2 + p_i^2)\right.\right.\right. \\
&\left.\left. - \frac{k\pi b_i}{2}\right)\right\} u(\mathbf{x})\Big|_{x_s=-\infty}^{x_s=\infty} - \int \frac{\partial}{\partial x_s}\left(exp\left\{\frac{f}{2b_i}\left(a_i x_i^2 - 2x_i(w_i - p_i) - 2w_i(d_i p_i - b_i q_i)\right.\right.\right. \\
&\left.\left.\left.\left. + d_i(w_i^2 + p_i^2) - \frac{k\pi b_i}{2}\right)\right\}\right) u(\mathbf{x}) dx_s\right] \prod_{j=k+1}^{n} \mathcal{K}^g_{A_j}(x_j, w_j) dx_1 dx_2 ... dx_s dx_{s+1} ... dx_n.
\end{aligned}$$

Hence for any $u \in \mathcal{S}(\mathbb{R}^n, \mathcal{C}l_{0,n})$ with $u(\mathbf{x})\Big|_{x_s=\pm\infty} = 0$, the above relation yields

$$\begin{aligned}
&\mathcal{O}_c\left[\frac{\partial u(\mathbf{x})}{\partial x_k}\right](w) \\
&= -\int_{\mathbb{R}^n} \frac{\partial}{\partial x_s}\left[\prod_{i=1}^{k} \frac{1}{\sqrt{(2\pi)^k b_i}} exp\left\{\frac{f}{2b_i}\Big(a_i x_i^2 - 2x_i(w_i - p_i) - 2w_i(d_i p_i - b_i q_i)\right.\right. \\
&\left.\left. + d_i(w_i^2 + p_i^2) - \frac{k\pi b_i}{2}\Big)\right\} u(\mathbf{x}) dx_s\right] \prod_{j=k+1}^{n} \mathcal{K}_{A_j}^{g}(x_j, w_j) dx_1 dx_2 ... dx_s dx_{s+1} ... dx_n \\
&= -\int_{\mathbb{R}^n} \prod_{i=1}^{k} \frac{1}{\sqrt{(2\pi)^k b_i}} exp\left\{\frac{f}{2b_i}\left(a_i x_i^2 - 2x_i(w_i - p_i) - 2w_i(d_i p_i - b_i q_i) + d_i(w_i^2 + p_i^2)\right.\right. \\
&\left.\left. - \frac{k\pi b_i}{2}\right)\right\}\left(\frac{f a_s x_s}{b_s} - \frac{f(w_s - p_s)}{b_s}\right) u(\mathbf{x}) \prod_{j=k+1}^{n} \mathcal{K}_{A_j}^{g}(x_j, w_j) dx_1 dx_2 ... dx_s dx_{s+1} ... dx_n \\
&= \int_{\mathbb{R}^n} \prod_{i=1}^{k} \mathcal{K}_{A_i}^{f}(x_i, w_i)\left(\frac{f(w_s - p_s)}{b_s} - \frac{f a_s x_s}{b_s}\right) u(\mathbf{x}) \prod_{i=1}^{k} \mathcal{K}_{A_j}^{g}(x_j, w_j) d^n\mathbf{x} \\
&= \frac{f}{b_s}\mathcal{O}_c\left[(w_s - p_s) u(\mathbf{x})\right](\mathbf{w}) - \frac{f a_s}{b_s}\mathcal{O}_c\left[x_s u(\mathbf{x})\right](\mathbf{w})
\end{aligned}$$

That proves our first part of the theorem.

Similarly for $k+1 \leq s \leq n$, we have

$$\mathcal{O}_c\left[\frac{\partial u(\mathbf{x})}{\partial x_k}\right](w) = \mathcal{O}_c\left[(w_s - p_s) u(\mathbf{x})\right](\mathbf{w})\frac{g}{b_s} - \mathcal{O}_c\left[x_s u(\mathbf{x})\right](\mathbf{w})\frac{g a_s}{b_s}$$

Hence the complete theorem is proved. □

Theorem 4.7. *Let $u \in \mathcal{S}(\mathbb{R}^n, \mathcal{C}l_{0,n})$ and $\mathcal{O}_c[u(\boldsymbol{x})](\boldsymbol{w})$ be its two-sided Clifford-valued offset linear canonical transform, then the following relations hold*

$$(i) \quad \frac{\partial}{\partial w_s}\left[\mathcal{O}_c[u(\boldsymbol{x})](\boldsymbol{w})\right] = \frac{f d_s}{b_s}\mathcal{O}_c\left[w_s u(\boldsymbol{x})\right](\boldsymbol{w}) - \frac{f}{b_s}\mathcal{O}_c\left[(d_s p_s - b_s q_s) u(\boldsymbol{x})\right](\boldsymbol{w}) - \frac{f}{b_s}\mathcal{O}_c\left[x_s u(\boldsymbol{x})\right](\boldsymbol{w}),$$

$1 \leq s \leq k$

$$(ii) \quad \frac{\partial}{\partial w_s}\left[\mathcal{O}_c[u(\boldsymbol{x})](\boldsymbol{w})\right] = \mathcal{O}_c\left[w_s u(\boldsymbol{x})\right](\boldsymbol{w})\frac{g d_s}{b_s} - \mathcal{O}_c\left[(d_s p_s - b_s q_s) u(\boldsymbol{x})\right](\boldsymbol{w})\frac{g}{b_s} - \mathcal{O}_c\left[x_s u(\boldsymbol{x})\right](\boldsymbol{w})\frac{g}{b_s}$$

$k+1 \leq s \leq n$

Proof. For $1 \leq s \leq k$, we have

$$\begin{aligned}
&\frac{\partial}{\partial w_s}\left[\mathcal{O}_c[u(\mathbf{x})](\mathbf{w})\right] \\
&= \frac{\partial}{\partial w_s}\left[\int_{\mathbb{R}^n} \prod_{i=1}^{k} \frac{1}{\sqrt{(2\pi)^k b_i}} exp\left\{\frac{f}{2b_i}\Big(a_i x_i^2 - 2x_i(w_i - p_i) - 2w_i(d_i p_i - b_i q_i) + d_i(w_i^2 + p_i^2)\right.\right. \\
&\left.\left. - \frac{k\pi b_i}{2}\Big)\right\} u(\mathbf{x}) \prod_{j=k+1}^{n} \mathcal{K}_{A_j}^{g}(x_j, w_j) d^n\mathbf{x}\right] \\
&= \int_{\mathbb{R}^n} \prod_{i=1}^{k} \frac{1}{\sqrt{(2\pi)^k b_i}} \frac{\partial}{\partial w_s} exp\left\{\frac{f}{2b_i}\Big(a_i x_i^2 - 2x_i(w_i - p_i) - 2w_i(d_i p_i - b_i q_i) + d_i(w_i^2 + p_i^2)\right. \\
&\left. - \frac{k\pi b_i}{2}\Big)\right\} u(\mathbf{x}) \prod_{j=k+1}^{n} \mathcal{K}_{A_j}^{g}(x_j, w_j) d^n\mathbf{x} \\
&= \int_{\mathbb{R}^n} \prod_{i=1}^{k} \frac{1}{\sqrt{(2\pi)^k b_i}} exp\left\{\frac{f}{2b_i}\Big(a_i x_i^2 - 2x_i(w_i - p_i) - 2w_i(d_i p_i - b_i q_i) + d_i(w_i^2 + p_i^2)\right.
\end{aligned}$$

$$- \frac{k\pi b_i}{2}\Big)\Big\} \left(\frac{-fx_s}{b_s} - \frac{f(d_sp_s - b_sq_s)}{b_s} + \frac{fw_sd_s}{b_s}\right)u(\mathbf{x}) \prod_{j=k+1}^{n} \mathcal{K}^g_{A_j}(x_j, w_j)d^n\mathbf{x}$$
$$= \int_{\mathbb{R}^n} \prod_{i=1}^{k} \mathcal{K}^f_{A_i}(x_i, w_i)\left(\frac{-fx_s}{b_s} - \frac{f(d_sp_s - b_sq_s)}{b_s} + \frac{fw_sd_s}{b_s}\right)u(\mathbf{x}) \prod_{j=k+1}^{n} \mathcal{K}^g_{A_j}(x_j, w_j)d^n\mathbf{x}$$
$$= \frac{fd_s}{b_s}\mathcal{O}_c\left[w_su(\mathbf{x})\right](\mathbf{w}) - \frac{f}{b_s}\mathcal{O}_c\left[(d_sp_s - b_sq_s)u(\mathbf{x})\right](\mathbf{w}) - \frac{f}{b_s}\mathcal{O}_c\left[x_su(\mathbf{x})\right](\mathbf{w}).$$

Similarly, for $k+1 \leq s \leq n$, we have

$$\frac{\partial}{\partial w_s}\left[\mathcal{O}_c[u(\mathbf{x})](\mathbf{w})\right]$$
$$= \frac{\partial}{\partial w_s}\Bigg[\int_{\mathbb{R}^n} \prod_{i=1}^{k} \mathcal{K}^f_{A_i}(x_i, w_i)u(\mathbf{x}) \prod_{j=k+1}^{n} \frac{1}{\sqrt{(2\pi)^{n-k}b_j}}$$
$$\times exp\left\{\frac{f}{2b_j}\left(a_jx_j^2 - 2x_j(w_j - p_j) - 2w_j(d_jp_j - b_jq_j) + d_j(w_j^2 + p_j^2) - \frac{(n-k)\pi b_j}{2}\right)\right\} d^n\mathbf{x}\Bigg].$$

After following the same steps as in the above case we have

$$\frac{\partial}{\partial w_s}\left[\mathcal{O}_c[u(\mathbf{x})](\mathbf{w})\right] = \mathcal{O}_c\left[w_su(\mathbf{x})\right](\mathbf{w})\frac{gd_s}{b_s} - \mathcal{O}_c\left[(d_sp_s - b_sq_s)u(\mathbf{x})\right](\mathbf{w})\frac{g}{b_s} - \mathcal{O}_c\left[x_su(\mathbf{x})\right](\mathbf{w})\frac{g}{b_s}$$

This completes the proof. □

Now we shall derive the novel convolution theorem for two-sided Clifford-valued special affine Fourier transform (3.1). To prove that we shall begin with following novel definition.

Definition 4.8. Let $u, v \in L^2(\mathbb{R}^n, \mathcal{C}l_{0,n})$, then the novel Clifford-valued convolution of u and v is defined by

$$u * v(\mathbf{x}) = \int_{\mathbb{R}^n} \prod_{i=1}^{k} \frac{1}{\sqrt{(2\pi)^k b_i}} exp\left\{\frac{f}{2b_i}\left(2a_i(y_i - x_i)y_i - 2w_i(d_ip_i - b_iq_i) + d_i(w_i^2 + p_i^2) - \frac{k\pi b_i}{2}\right)\right\}$$
$$\times u(\mathbf{y})v(\mathbf{x\text{-}y})$$
$$\times \prod_{j=k+1}^{n} \frac{1}{\sqrt{(2\pi)^{n-k}b_j}} exp\left\{\frac{f}{2b_j}\left(2a_j(y_j - x_j)y_j - 2w_j(d_jp_j - b_jq_j) + d_j(w_j^2 + p_j^2) - \right.\right.$$
$$\left.\left.\frac{(n-k)\pi b_j}{2}\right)\right\} d\mathbf{y}.$$

Now we shall investigate the convolution theorem associated with (3.1).

Theorem 4.9. *(Convolution Theorem)*
Let $\mathcal{O}_c[u(\boldsymbol{x})](\boldsymbol{w})$ and $\mathcal{O}_c[v(\boldsymbol{x})](\boldsymbol{w})$ be the two-sided Clifford-valued special affine Fourier transforms of u and v respectively. Then we have the following result

$$\mathcal{O}_c\left[u * v(\boldsymbol{x})\right](\boldsymbol{w}) = \mathcal{O}_c[u(\boldsymbol{x})](\boldsymbol{w})\mathcal{O}_c[v(\boldsymbol{x})](\boldsymbol{w})$$

Proof. By using the Definitions 3.1 and 4.8, we have

$$\mathcal{O}_c\left[u * v(\mathbf{x})\right](\mathbf{w})$$
$$= \int_{\mathbb{R}^n} \prod_{i=1}^{k} \mathcal{K}^f_{A_i}(x_i, w_i)\left[u * v(\mathbf{x})\right] \prod_{j=k+1}^{n} \mathcal{K}^g_{A_j}(x_j, w_j)d\mathbf{x}$$
$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \prod_{i=1}^{k} \mathcal{K}^f_{A_i}(x_i, w_i) \prod_{i=1}^{k} \frac{1}{\sqrt{(2\pi)^k b_i}} exp\left\{\frac{f}{2b_i}\left(2a_i(y_i - x_i)y_i - 2w_i(d_ip_i - b_iq_i) + d_i(w_i^2 + p_i^2)\right.\right.$$
$$\left.\left.-\frac{k\pi b_i}{2}\right)\right\} u(\mathbf{y})v(\mathbf{x\text{-}y})$$
$$\times \prod_{j=k+1}^{n} \frac{1}{\sqrt{(2\pi)^{n-k}b_j}} exp\left\{\frac{f}{2b_j}\left(2a_j(y_j - x_j)y_j - 2w_j(d_jp_j - b_jq_j) + d_j(w_j^2 + p_j^2) - \right.\right.$$

$$\left.\left.\frac{(n-k)\pi b_j}{2}\right)\right\} d\mathbf{y} \prod_{j=k+1}^{n} \mathcal{K}_{A_j}^{g}(x_j, w_j) d\mathbf{x}$$

$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\prod_{i=1}^{k}\frac{1}{\sqrt{(2\pi)^k}b_i}exp\left\{\frac{f}{2b_i}\left(a_i x_i^2 - 2x_i(w_i-p_i) - 2w_i(d_ip_i - b_iq_i) + d_i(w_i+p_i^2) - \frac{k\pi b_i}{2}\right)\right\}$$

$$\times\prod_{i=1}^{k}\frac{1}{\sqrt{(2\pi)^k}b_i}exp\left\{\frac{f}{2b_i}\left(2a_i(y_i-x_i)y_i - 2w_i(d_ip_i - b_iq_i) + d_i(w_i^2+p_i^2) - \frac{k\pi b_i}{2}\right)\right\}$$

$$\times u(\mathbf{y})v(\mathbf{x}\text{-}\mathbf{y})$$

$$\times\prod_{j=k+1}^{n}\frac{1}{\sqrt{(2\pi)^{n-k}}b_j}exp\left\{\frac{f}{2b_j}\left(2a_j(y_j-x_j)y_j - 2w_j(d_jp_j - b_jq_j) + d_j(w_j^2+p_j^2) -\right.\right.$$

$$\left.\left.\frac{(n-k)\pi b_j}{2}\right)\right\} d\mathbf{y}\prod_{j=k+1}^{n}\frac{1}{\sqrt{(2\pi)^{n-k}}b_j}exp\left\{\frac{f}{2b_j}\left((a_jx_j^2 - 2x_j(w_j-p_j) - 2w_j(d_jp_j - b_jq_j)\right.\right.$$

$$\left.\left.+d_j(w_j^2+p_j^2) - \frac{(n-k)\pi b_j}{2}\right)\right\} d\mathbf{x}.$$

Set **x-y=z** for all i, j

$$\mathcal{O}_c\left[u * v(\mathbf{x})\right](\mathbf{w})$$

$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(\mathbf{y})v(\mathbf{z})\prod_{i=1}^{k}\frac{1}{\sqrt{(2\pi)^k}b_i}exp\left\{\frac{f}{2b_i}\left(a_i(y_i+z_i)^2 - 2(y_i+z_i)(w_i-p_i) - 2w_i(d_ip_i - b_iq_i)\right.\right.$$

$$\left.\left.+d_i(w_i+p_i^2) - \frac{k\pi b_i}{2}\right)\right\}$$

$$\times\prod_{i=1}^{k}\frac{1}{\sqrt{(2\pi)^k}b_i}exp\left\{\frac{f}{2b_i}\left(2a_i(-z_i)y_i - 2w_i(d_ip_i - b_iq_i) + d_i(w_i^2+p_i^2) - \frac{k\pi b_i}{2}\right)\right\}$$

$$\times\prod_{j=k+1}^{n}\frac{1}{\sqrt{(2\pi)^{n-k}}b_j}exp\left\{\frac{f}{2b_j}\left(2a_j(-z_j)y_j - 2w_j(d_jp_j - b_jq_j) + d_j(w_j^2+p_j^2)\right.\right.$$

$$\left.\left.-\frac{(n-k)\pi b_j}{2}\right)\right\}\prod_{j=k+1}^{n}\frac{1}{\sqrt{(2\pi)^{n-k}}b_j}exp\left\{\frac{f}{2b_j}\left(a_j(y_j+z_j)^2 - 2(y_j+z_j)(w_j-p_j)\right.\right.$$

$$\left.\left.-2w_j(d_jp_j - b_jq_j) + d_j(w_j^2+p_j^2) - \frac{(n-k)\pi b_j}{2}\right)\right\} d\mathbf{z}d\mathbf{y}$$

$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(\mathbf{y})v(\mathbf{z})\prod_{i=1}^{k}\frac{1}{\sqrt{(2\pi)^k}b_i}exp\left\{\frac{f}{2b_i}\left(a_i(y_i^2+z_i^2+2y_iz_i) - 2(y_i+z_i)(w_i-p_i)\right.\right.$$

$$\left.\left.-2w_i(d_ip_i - b_iq_i) + d_i(w_i+p_i^2) - \frac{k\pi b_i}{2}\right)\right\}$$

$$\times\prod_{i=1}^{k}\frac{1}{\sqrt{(2\pi)^k}b_i}exp\left\{\frac{f}{2b_i}\left(2a_i(-z_i)y_i - 2w_i(d_ip_i - b_iq_i) + d_i(w_i^2+p_i^2) - \frac{k\pi b_i}{2}\right)\right\}$$

$$\times\prod_{j=k+1}^{n}\frac{1}{\sqrt{(2\pi)^{n-k}}b_j}exp\left\{\frac{f}{2b_j}\left(2a_j(-z_j)y_j - 2w_j(d_jp_j - b_jq_j) + d_j(w_j^2+p_j^2)\right.\right.$$

$$\left.\left.-\frac{(n-k)\pi b_j}{2}\right)\right\}\prod_{j=k+1}^{n}\frac{1}{\sqrt{(2\pi)^{n-k}}b_j}exp\left\{\frac{f}{2b_j}\left(a_j(y_j^2+z_j^2+2y_jz_j) - 2(y_j+z_j)(w_j-p_j)\right.\right.$$

$$\left.\left.-2w_j(d_jp_j - b_jq_j) + d_j(w_j^2+p_j^2) - \frac{(n-k)\pi b_j}{2}\right)\right\} d\mathbf{z}d\mathbf{y}$$

On further simplifying we get,

$$\mathcal{O}_c\left[u * v(\mathbf{x})\right](\mathbf{w})$$

$$= \int_{\mathbb{R}^n}\prod_{i=1}^{k}\frac{1}{\sqrt{(2\pi)^k}b_i}exp\left\{\frac{f}{2b_i}\left(a_iy_i^2 - 2y_i(w_i-p_i) - 2w_i(d_ip_i - b_iq_i) + d_i(w_i^2+p_i^2) - \frac{k\pi b_i}{2}\right)\right\}$$

$$\times u(\mathbf{y})\int_{\mathbb{R}^n}\prod_{j=k+1}^{n}\frac{1}{\sqrt{(2\pi)^{n-k}b_j}}exp\left\{\frac{g}{2b_j}\left(a_jy_j^2-2y_j(w_j-p_j)-2w_j(d_jp_j-b_jq_j)+d_j(w_j^2+p_j^2)\right.\right.$$
$$\left.\left.-\frac{(n-k)\pi b_j}{2}\right)\right\}d\mathbf{y}$$
$$\times\int_{\mathbb{R}^n}\prod_{i=1}^{k}\frac{1}{\sqrt{(2\pi)^k b_i}}exp\left\{\frac{f}{2b_i}\left(a_iz_i^2-2z_i(w_i-p_i)-2w_i(d_ip_i-b_iq_i)+d_i(w_i^2+p_i^2)-\frac{k\pi b_i}{2}\right)\right\}$$
$$\times v(\mathbf{y})\int_{\mathbb{R}^n}\prod_{j=k+1}^{n}\frac{1}{\sqrt{(2\pi)^{n-k}b_j}}exp\left\{\frac{g}{2b_j}\left(a_jz_j^2-2z_j(w_j-p_j)-2w_j(d_jp_j-b_jq_j)+d_j(w_j^2+p_j^2)\right.\right.$$
$$\left.\left.-\frac{(n-k)\pi b_j}{2}\right)\right\}d\mathbf{z}$$
$$=\mathcal{O}_c[u(\mathbf{x})](\mathbf{w})\mathcal{O}_c[v(\mathbf{x})](\mathbf{w}).$$

This completes the proof. □

5. Conclusion

In this article, we have derived the following objectives: First, we have introduced the novel transform, that is Clifford valued special affine Fourier transform and studied its fundamental properties along with Plancherelformula. Secondly, we studied boundedness, continuity and differentiation theorems for the proposed transform. Finally we have introduced a novel Clifford convolution and the corresponding theorem has been also derived.

Competing interests. The authors declare no competing interests.

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