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## Quadratic phase S-Transform: Properties and uncertainty principles

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## ABSTRACT

In this paper, a novel quadratic phase S-transform (QPST) is proposed, by generalizing the S-transform (ST) with five parameters a, b, c,d and e. QPST displays the time and quadratic phase domain-frequency information jointly in the time-frequency plane. Firstly, we define the novel QPST and give its relation with quadratic phase Fourier transform (QPFT). Secondly, several important properties of newly defined QPST, such as conjugation, translation, modulation, orthogonality relation and reconstruction formula are derived. Finally, we formulate several classes of uncertainty inequalities, such as the Heisenberg uncertainty inequality and logarithmic uncertainty inequality.

## 1. Introduction

To avail the local characteristics of non-stationary signals the timefrequency analysis tools viz : short-time Fourier transform (STFT) [1], Wigner-Ville distribution [2], Ambiguity function and wavelet transform (WT) [3] plays an important role. However these tools have a major drawback as they suffer from cross-term interference, low resolution and other issues. In order to overcome these drawbacks a series of signal processing tools have been proposed including S-transform (ST) [4], the fractional Fourier transform (FRFT) [5], the fractional S-transform (FRST) [6], the linear canonical transform (LCT) [7], the linear canonical S-transform (LCST) [8], and so on [9–15].

In 1996 R.G. Stockwell et al. [16] proposed S-transform by combining the merits wavelet transform and short-time Fourier transform, with moving and scalable window function. The 1-D continuous S-transform of any signal f(x) is given as [17]

$$\mathscr{S}_{q}f(\xi,w) = \int_{\mathbb{R}} f(t)\varphi(\xi-x,w)e^{-2\pi iwx}dx,$$
(1.1)

where  $\varphi$  is a window function and is taken so that  $\int_{\mathbb{R}} \varphi(x, w) dx = 1$  for every non-zero real number w. In the context of time-frequency analysis, the S-transform provides more exact local characteristics of nonstationary signals. ST has gained much popularity in last decades because of its applications in several fields of engineering, geo-physis, optics, bio-informatics, gravitational waves, and signal processing in general [18–22]. Although ST is one of the promising tool for the analysis of non-stationary signals, but it is incompetent for the analysis of chirp like signals whose energy is not well concentrated in the frequency domain.

The quadratic phase Fourier transform (QPFT)which is a neoteric addition to the class of Fourier transforms and embodies a variety of signal processing tools including the Fourier, fractional Fourier, linear canonical and special affine Fourier transform. QPFT provides a unified treatment of both the transient and non-transient signals in a simple and insightful fashion. The QPFT was first introduced by Castro et al. [23] as:

$$\mathscr{Q}_{\Omega}[f](w) = \int_{\mathbb{R}} f(x) \mathscr{H}_{\Omega}(x, w) dx$$
(1.2)

where  $\mathcal{H}_{\Omega}(x,w)$  denotes the quadratic-phase Fourier kernel and is given by

$$\mathscr{H}_{\Omega}(x,w) = \frac{1}{\sqrt{2\pi}} e^{i\left(ax^2 + bxw + cw^2 + dx + ew\right)}.$$
(1.3)

where  $a, b, c, d, e \in \mathbb{R}$ ,  $b \neq 0$ . These arbitrary real parameters present in (1.3) are of great importance as their choice sense of rotation as well as shift can be inculcated in both the axis of time and frequency domain. Hence can be used in better analysis of non-stationary signals which are

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employed in radar and other communication systems. Due to its global kernel and extra degrees of freedom, the QPFT has arrived an efficient tool in solving several problems arising in diverse branches of science and engineering, including harmonic analysis, image processing, sampling, reproducing kernel Hilbert spaces and so on [24–26]. However, due its global kernel the QPFT can not reveal quadractic phase spectrum of some non-stationary signals. Recently, different generalized versions of QPFT have been introduced to improve the performance in concentration [see [27–31]].

Keeping in view the fascinating applications of the S-transforms the authors generalize ST to various novel integral transforms including linear canonical S-transform, fractional S-transform, and so on [32–34]. They have attained a much more attention of the engineering and signal processing community. But to the best of our knowledge theory about quadratic-phase S-transform (QPST) have never been proposed up to date, therefore it is worthwhile to study the theory of QPST based on the ST and QPFT, which can be productive for signal processing theory and applications. Therefore, the cynosure of this paper is to rigorously study the QPST.

## 1.1. Paper contributions

The contributions of this paper are summarized below:

- We introduce a novel integral transform coined as the quadratic phase S-transform.
- We establish the fundamental relationship between the proposed transform (QPST) and the quadratic phase Fourier transform (QPFT).
- To study the fundamental properties of the proposed transform, including the conjugation, translation, modulation, orthogonality relation and inversion formula.
- To formulate several classes of uncertainty principles, such as the Heisenberg uncertainty principle and the logarithmic uncertainty principle associated with the quadratic phase S-transform.

### 1.2. Paper outlines

The paper is organized as follows: In Section 2, we introduce the novel quadractic phase S- transform and present its relation with quadractic phase Fourier transform, then several important properties of the novel QPST, including translation, modulation, orthogonality relation and reconstruction formula are derived. In Section 3, we establish

Heisenberg-type uncertainty principle and logarithmic uncertainty principle associated with the proposed QPST. Finally, a conclusion is drawn in Section 4.

## 2. Quadratic phase S-Transform

In this section we define the novel quadratic phase S-transform (QPST) which is an extension of S-transform and quadratic phase Fourier transform.

**Definition 2.1.** Let  $\varphi$  be a non-zero window function in  $L^2(\mathbb{R})$  and  $\Omega = (a, b, c, d, e), b \neq 0$  be a given parameter set, then the quadratic phase S-transform of any signal  $f \in L^2(\mathbb{R})$  with respect to  $\varphi$  is denoted by  $\mathscr{S}^{\Omega}_{\omega} f(\xi, w)$  and defined as

$$\mathscr{S}^{\Omega}_{\varphi}f(\xi,w) = \int_{\mathbb{R}} f(x)\overline{\varphi(\xi-x,w)} \mathscr{K}_{\Omega}(x,w) dx$$
(2.1)

where  $\mathscr{K}_{\Omega}(x, w)$  is given in (1.3).

**Remark 2.1.** By varying the parameter  $\Omega = (a, b, c, d, e)$ , Definition 2.1 gives birth to some of the prominent integral transforms as indicated below:

• For  $\Omega = (a/2b, -1/b, c/2b, 0, 0)$  and multiplying the RHS of (2.1) by  $1/\sqrt{ib}$ , Definition 2.1 boils down to the linear canonical S-transform [8]:

$$\mathscr{S}^{\Omega}_{\varphi}f(\xi,w) = \frac{1}{\sqrt{2\pi ib}} \int_{\mathbb{R}} f(x)\overline{\varphi(\xi-x,w)} e^{i\frac{1}{2b}\left(ax^2 - 2xw + cw^2\right)} dx.$$
(2.2)

For Ω = (cotα/2, -cscα/2, cotθ/2, 0, 0), α ≠ nπ and multiplying the RHS of (2.1) by √1 − *i*cotα, Definition 2.1 yields the fractional S-transform [6]:

$$\mathscr{S}^{\Omega}_{\varphi}f(\xi,w) = \sqrt{\frac{1-i\cot\alpha}{2\pi}} \int_{\mathbb{R}} f(x)\overline{\varphi(\xi-x,w)} e^{\frac{i}{2}\left(x^2+w^2\right)\cot\alpha-iwx\csc\alpha} dx.$$
(2.3)

For Ω = (0, 1, -1, 0, 0), Definition 2.1 yields the classical S-transform [16]. Now we establish the fundamental relationship between QPST and the QPFT as:

$$\mathscr{S}^{\Omega}_{\varphi}f(\xi,w) = \mathscr{Q}_{\Omega}[f(x)\overline{\phi(\xi-x,w)}](w,\xi)$$
(2.4)

Applying inverse QPFT, we get

$$f(x)\overline{\phi(\xi-x,w)} = \mathscr{Q}_{-\Omega}\left[\mathscr{S}^{\Omega}_{\varphi}f(\xi,w)\right]$$

$$= \frac{|b|}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathscr{S}^{\Omega}_{\varphi}f(\xi,w) e^{-i\left(ax^{2}+bxw+cw^{2}+dx+ew\right)}dw.$$
(2.5)

Moreover, if  $\varphi \in L^1 \cap L^2(\mathbb{R})$  satisfies  $\int_{\mathbb{R}} \overline{\varphi(\xi - x, w)} d\xi = 1$  then for every  $f \in L^2(\mathbb{R})$ , the QPST defined in (2.1) reduces to the QPFT (1.2) as follows:

$$\begin{split} \int_{\mathbb{R}} \mathscr{S}^{\Omega}_{\varphi} f(\xi, w) d\xi &= \int_{\mathbb{R}^2} f(x) \overline{\varphi(\xi - x, w)} \mathscr{K}_{\Omega}(x, w) d\xi dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{i\left(ax^2 + bxw + cw^2 + dx + ew\right)} dx \int_{\mathbb{R}} \overline{\varphi(\xi - x, w)} d\xi \\ &= \mathscr{C}_{\Omega}[f](w). \end{split}$$

Prior to establishing the vital properties of the proposed QPST, we present an explicit example for lucid illustration of the proposed Definition 2.1:

Consider the Gaussian function  $f(x) = e^{-iax^2}$ , then the QPST of f(x) with respect to the real parameter set  $\Omega = (a, b, c, d, e), b \neq 0$  and the normalized window function

$$\varphi(x) = \begin{cases} 1, & \text{if } -\frac{1}{2} \le x_1 \le \frac{1}{2}, \\ 0, & \text{elsewhere} \end{cases}$$
(2.6)

can be computed as

$$\mathcal{S}^{\Omega}_{\varphi}f(\xi,w) = \frac{1}{\sqrt{2\pi}} \int_{v-\frac{1}{2}}^{v+\frac{1}{2}} e^{i(bxw+cw^2+dx+ew)} dx$$
$$= \frac{e^{icw^2+iew}}{\sqrt{2\pi}} \int_{v-\frac{1}{2}}^{v+\frac{1}{2}} e^{i(bw+d)x} dx$$
$$= \frac{e^{i(bwv+cw^2+dv+ew)}}{(bw+d)\sqrt{2\pi}} \left( e^{\frac{(bw+d)}{2}} - e^{-\frac{(bw+d)}{2}} \right)$$

We are now ready to investigate some of the fundamental properties of the proposed QPST. In this direction, we have the following theorem which assembles some of the basic properties of the proposed QPST defined in (2.1).

**Theorem 2.2.** Let  $\varphi, f \in L^2(\mathbb{R})$ , where  $\phi$  is a window function, then quadratic phase S-transform defined in (2.1) satisfies the following properties: ~

(i)Linearity: 
$$\mathscr{S}^{\Omega}_{\varphi}[\alpha f + \beta g](\xi, w) = \alpha \mathscr{S}^{\Omega}_{\varphi}f(\xi, w) + \beta \mathscr{S}^{\Omega}_{\varphi}g(\xi, w).$$
  
(ii)Conjugation:  $\mathscr{S}_{\overline{\varphi}}^{\Omega}\overline{f}(\xi, w) = \overline{\mathscr{S}^{\Omega}_{\varphi}}f(\xi, w).$   
(iii)Pairity:  $\mathscr{S}^{\Omega}_{P\varphi}[Pf](\xi, w) = \mathscr{S}^{\Omega'}_{\varphi}[f](-\xi, -w), Pf(x) = f(-x) \text{ and } \Omega' = (a, b, c, -d, -e).$   
(iv)Translation:  $\mathscr{S}^{\Omega}_{\varphi}[\tau_{x_0}f](\xi, w) = e^{i(\alpha x_0^2)}$ 

) Translation: 
$$\mathscr{S}^{ss}_{\varphi}[\tau_{x_0}f](\xi, w) = e^{i(dx_0)}$$

$$+bwx_0+dx_0)\mathscr{S}^{\Omega}_{\varphi}[F](\xi-x_0,w),$$

 $\tau_{x_0} f(x) = f(x - x_0)$  and  $F(t) = e^{i2ax_0t} f(t)$ . (v)Modulation:  $\mathscr{S}^{\Omega}_{\omega}[\mathscr{M}_{w_0}f](\xi,w)$ 

$$=e^{-i\left[\frac{c}{b^2}w_0^2+2\frac{c}{b}w_0w+\frac{c}{b}w_0\right]}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}f(x)\overline{\varphi(\xi-x,w)}\mathscr{H}_{\Omega}\left(x,w+\frac{w_0}{b}\right)dx,$$

 $\mathscr{M}_{w_0}f(\mathbf{x}) = e^{iw_0x}f(\mathbf{x}).$ 

**Proof.** (i) We omit proof of (i) as it is a direct consequence of the Definition 2.1.

(ii) From (2.1), we have

$$\begin{split} \mathscr{S}_{\overline{\varphi}}^{\alpha} \widehat{f}(\xi,w) &= \int_{\mathbb{R}} \widehat{f(x)} \overline{\varphi}(\xi-x,w) \mathscr{K}_{\Omega}(x,w) dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \overline{f(x)} \overline{\varphi}(\xi-x,w) e^{i\left(ax^{2}+bxw+cw^{2}+dx+ew\right)} dx \\ &= \int_{\mathbb{R}} \overline{f(x)} \overline{\varphi(\xi-x,w)} e^{-i(ax^{2}+bxw+cw^{2}+dx+ew)} dx \\ &= \overline{\int_{\mathbb{R}} f(x)} \overline{\varphi(\xi-x,w)} \mathscr{K}_{-\Omega}(x,w) dx \\ &= \overline{\int_{\mathbb{R}} f(x)} \overline{\varphi(\xi-x,w)} \mathscr{K}_{-\Omega}(x,w) dx \end{split}$$

(iii) From the Definition 2.1, It follows

$$\begin{aligned} \mathscr{S}^{\Omega}_{P\varphi}[Pf](\xi,w) \\ &= \int_{\mathbb{R}} Pf(x)\overline{P\varphi(\xi-x,w)}\mathscr{K}_{\Omega}(x,w)dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(-x)\overline{\varphi(-\xi-(-x),-w)}e^{i\left(ax^{2}+bxw+cw^{2}+dx+ew\right)}dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(-x)\overline{\varphi(-\xi-(-x),-w)}e^{i\left[a(-x)^{2}+b(-x)(-w)+c(-w)^{2}+(-d)(-x)+(-e)(-w)\right]}dx \\ &= \mathscr{S}^{\Omega'}_{\varphi}[f](-\xi,-w), \text{ where } \Omega' = (a,b,c,-d,-e). \end{aligned}$$

(iv) Let  $\varphi, f \in L^2(\mathbb{R})$ , then we have

$$\begin{aligned} \mathscr{S}^{\Omega}_{\varphi} \Big[ \tau_{x_{0}} f \Big] (\xi, w) \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x - x_{0}) \overline{\varphi(\xi - x, w)} e^{i \left(ax^{2} + bxw + cw^{2} + dx + ew\right)} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) \overline{\varphi(\xi - (t + x_{0}), w)} e^{i \left[a(t + x_{0})^{2} + b(t + x_{0})w + cw^{2} + d(t + x_{0}) + ew\right]} dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) \overline{\varphi(\xi - x_{0}) - t, w} e^{i \left(ax^{2} + btw + cw^{2} + dt + ew\right)} \\ &\qquad \times e^{i \left(ax_{0}^{2} + 2ax_{0}t + bwx_{0} + dx_{0}\right)} dt \\ &= e^{i \left(ax_{0}^{2} + bwx_{0} + dx_{0}\right)} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i 2ax_{0}t} f(t) \overline{\varphi(\xi - x_{0}) - t, w)} e^{i \left(at^{2} + btw + cw^{2} + dt + ew\right)} dt \\ &= e^{i \left(ax_{0}^{2} + bwx_{0} + dx_{0}\right)} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} F(t) \overline{\varphi(\xi - x_{0}) - t, w)} e^{i \left(at^{2} + btw + cw^{2} + dt + ew\right)} dt \\ &= e^{i \left(ax_{0}^{2} + bwx_{0} + dx_{0}\right)} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} F(t) \overline{\varphi(\xi - x_{0}) - t, w)} e^{i \left(at^{2} + btw + cw^{2} + dt + ew\right)} dt \\ &= e^{i \left(ax_{0}^{2} + bwx_{0} + dx_{0}\right)} \mathscr{S}^{\Omega}_{a} [F](\xi - x_{0}, w). \end{aligned}$$

(v) From the Definition of QPST, we get

This completes the proof of the theorem.  $\Box$ 

**Theorem 2.3.** (Orthogonality relation) Let  $\mathscr{S}^{\Omega}_{\varphi}f_1(\xi, w)$  and  $\mathscr{S}^{\Omega}_{\varphi}f_2(\xi, w)$  be the quadratic phase S-transform of the signals  $f_1, f_2 \in L^2(\mathbb{R})$ , respectively. Then

$$\int_{\mathbb{R}^2} \mathscr{S}^{\Omega}_{\varphi} f_1(\xi, w) \overline{\mathscr{S}^{\Omega}_{\varphi} f_2(\xi, w)} dw d\xi = \frac{1}{b} \left\langle f_1 \int_{\mathbb{R}} |\varphi(\xi, w)|^2 d\xi, f_2 \right\rangle.$$
(2.7)

### Proof. From definition of QPST, we conclude that

$$f_1 = \frac{b}{\mathscr{C}_{\phi}\sqrt{2\pi}} \int_{\mathbb{R}^2} \mathscr{I}_{\varphi}^{\Omega} f_1(\xi, w) \varphi(\xi - x, w) e^{-i\left(ax^2 + bxw + cw^2 + dx + ew\right)} d\xi dw.$$
(2.12)

Which completes the proof.

**Remark 2.4**. If we take  $f_1 = f_2 = f$ , the Theorem 2.3 yields

$$\int_{\mathbb{R}^2} |\mathscr{S}^{\Omega}_{\varphi} f(\xi, w)|^2 dw d\xi = \frac{1}{b} ||f||^2 \int_{\mathbb{R}} |\varphi(\xi, w)|^2 d\xi.$$
(2.8)

**Theorem 2.5.** (*Reconstruction formula*) Every signal  $f \in L^2(\mathbb{R})$  can be recovered back from the quadratic phase S-transform by the formula

$$f(x) = \frac{b}{\mathscr{C}_{\phi}\sqrt{2\pi}} \int_{\mathbb{R}^2} \mathscr{S}^{\Omega}_{\varphi} f_1(\xi, w) \varphi(\xi - x, w) e^{-i\left(ax^2 + bxw + cw^2 + dx + ew\right)} d\xi dw, \qquad (2.9)$$

where  $\mathscr{C}_{\phi} = \int_{\mathbb{R}} |\varphi(\xi, w)|^2 d\xi$ .

**Proof.** From Theorem 2.3,we have

$$\begin{split} &\frac{1}{b} \left\langle f_{1} \int_{\mathbb{R}} |\varphi(\xi,w)|^{2} d\xi, f_{2} \right\rangle \\ &= \int_{\mathbb{R}^{2}} \mathscr{S}_{\varphi}^{\Omega} f_{1}(\xi,w) \overline{\mathscr{S}_{\varphi}^{\Omega} f_{2}(\xi,w)} dw d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^{2}} \mathscr{S}_{\varphi}^{\Omega} f_{1}(\xi,w) \int_{\mathbb{R}} \overline{f_{2}(x)} \varphi(\xi-x,w) e^{-i\left(ax^{2}+bxw+cw^{2}+dx+ew\right)} dx d\xi dw \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^{3}} \mathscr{S}_{\varphi}^{\Omega} f_{1}(\xi,w) \varphi(\xi-x,w) e^{-i\left(ax^{2}+bxw+cw^{2}+dx+ew\right)} \overline{f_{2}(x)} dx d\xi dw \\ &= \left\langle \int_{\mathbb{R}^{2}} \mathscr{S}_{\varphi}^{\Omega} f_{1}(\xi,w) \varphi(\xi-x,w) \frac{1}{\sqrt{2\pi}} e^{-i\left(ax^{2}+bxw+cw^{2}+dx+ew\right)} d\xi dw, f_{2}(x) \right\rangle. \end{split}$$

$$(2.10)$$

As (2.10) is valid for all  $f_2 \in L^2(\mathbb{R})$ , we have

$$f_{1} \int_{\mathbb{R}} |\varphi(\xi, w)|^{2} d\xi = \frac{b}{\sqrt{2\pi}} \int_{\mathbb{R}^{2}} \mathscr{S}^{\Omega}_{\varphi} f_{1}(\xi, w) \varphi(\xi - x, w) e^{-i\left(ax^{2} + bxw + cw^{2} + dx + ew\right)} d\xi dw$$
(2.11)

Equivalently,

Which completes the proof.  $\Box$ 

## 3. Uncertainty principle

In this section based on the fundamental relationship between QPST and QPFT, we investigate Heisenberg uncertainty principle and logarithmic uncertainty principle associated with the proposed QPST.

**Theorem 3.1.** (QPFT Heisenberg inequality [27]) Let  $\mathscr{C}_{\Omega}[f]$  be the quadratic-phase Fourier transform of any signal  $f \in L^2(\mathbb{R})$ , then the following inequality holds:

$$\int_{\mathbb{R}} w^2 |\mathscr{Q}_{\Omega}[f](w)|^2 dw \int_{\mathbb{R}} x^2 |f(x)|^2 dx \ge \frac{1}{4b^2} \left( \int_{\mathbb{R}} |f(x)|^2 dx \right)^2.$$
(3.1)

Above equation can be rewritten as

$$\int_{\mathbb{R}} w^2 |\mathscr{Q}_{\Omega}[f](w)|^2 dw \int_{\mathbb{R}} x^2 |\mathscr{Q}_{\Omega}^{-1} \{\mathscr{Q}_{\Omega}[f]\}(x)|^2 dx \ge \frac{1}{4b^2} \left( \int_{\mathbb{R}} |f(x)|^2 dx \right)^2.$$
(3.2)

Or, equivalently,

$$\int_{\mathbb{R}} w^2 |\mathscr{Q}_{\Omega}[f](w)|^2 dw \int_{\mathbb{R}} x^2 |\mathscr{Q}_{\Omega}^{-1} \{\mathscr{Q}_{\Omega}[f]\}(x)|^2 dx \ge \left(\frac{1}{2} \int_{\mathbb{R}} |\mathscr{Q}_{\Omega}[f](w)|^2 dw\right)^2.$$
(3.3)

**Lemma 3.1.** Let  $\mathscr{S}^\Omega_{\varphi} f$  be the quadratic phase S-transform of the signal  $f \in L^2(\mathbb{R})$ , then

$$\mathscr{C}_{\varphi} \int_{\mathbb{R}} x^2 |f(x)|^2 dx = \frac{1}{b^2} \int_{\mathbb{R}} \int_{\mathbb{R}} x^2 \left| \mathscr{C}_{\Omega}^{-1} \left\{ \mathscr{S}_{\varphi}^{\Omega} f(\xi, w) \right\}(x) \right|^2 dx d\xi,$$
(3.4)

where  $\mathscr{C}_{\varphi} = \int_{\mathbb{R}} |\varphi(\xi, w)|^2 d\xi$ 

**Proof.** For the sake of brevity we omit proof.  $\Box$ 

**Theorem 3.2.** (QPST Heisenberg inequality) Let  $\mathscr{S}^{\Omega}_{\varphi} f$  be the quadratic phase S-transform of any signal  $f \in L^2(\mathbb{R})$  with respect  $\varphi \in L^2(\mathbb{R})$ , then we

have the following inequality:

$$\left(\int_{\mathbb{R}}\int_{\mathbb{R}}w^{2}\left|\mathscr{S}_{\varphi}^{\Omega}f(\xi,w)\right|^{2}dwd\xi\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}x^{2}\left|f(x)\right|^{2}dx\right)^{\frac{1}{2}}\geq\frac{\sqrt{\mathscr{C}_{\varphi}}}{2b^{2}}\|f\|^{2}.$$
(3.5)

**Proof.** As both  $\mathscr{C}_{\Omega}[f]$  and  $\mathscr{S}_{\varphi}^{\Omega}f$  are in  $L^{2}(\mathbb{R})$ , therefore replacing  $\mathscr{C}_{\Omega}[f]$  by  $\mathscr{S}_{\varphi}^{\Omega}f$  in (3.3), we get

$$\int_{\mathbb{R}} w^{2} |\mathscr{S}_{\varphi}^{\Omega} f(\xi, w)|^{2} dw \int_{\mathbb{R}} x^{2} |\mathscr{C}_{\Omega}^{-1} \{\mathscr{S}_{\varphi}^{\Omega} f(\xi, w)\}(x)|^{2} dx$$

$$\geq \left(\frac{1}{2} \int_{\mathbb{R}} |\mathscr{S}_{\varphi}^{\Omega} f(\xi, w)|^{2} dw\right)^{2}.$$
(3.6)

Now integrating (3.6) with respect to  $d\xi$ , we obtain

$$\begin{split} \int_{\mathbb{R}} & \left\{ \left( \int_{\mathbb{R}} w^2 \left| \mathscr{S}^{\Omega}_{\varphi} f(\xi, w) \right|^2 dw \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}} x^2 \left| \mathscr{C}^{-1}_{\Omega} \left\{ \mathscr{S}^{\Omega}_{\varphi} f(\xi, w) \right\}(x) \right|^2 dx \right)^{\frac{1}{2}} \right\} d\xi \\ & \geq \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \mathscr{S}^{\Omega}_{\varphi} f(\xi, w) \right|^2 dw d\xi. \end{split}$$

$$(3.7)$$

Applying Cauchy-Schwartz inequality to the LHS of (3.7), it yields

$$\begin{split} \left( \int_{\mathbb{R}} \int_{\mathbb{R}} w^{2} \left| \mathscr{S}_{\varphi}^{\Omega} f(\xi, w) \right|^{2} dw d\xi \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}} \int_{\mathbb{R}} x^{2} \left| \mathscr{C}_{\Omega}^{-1} \left\{ \mathscr{S}_{\varphi}^{\Omega} f(\xi, w) \right\}(x) \right|^{2} dx d\xi \right)^{\frac{1}{2}} \\ &\geq \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \mathscr{S}_{\varphi}^{\Omega} f(\xi, w) \right|^{2} dw d\xi. \end{split}$$

$$(3.8)$$

Applying Lemma 3.1 to LHS and (2.8) to RHS, (3.8) yields

$$\left(\int_{\mathbb{R}}\int_{\mathbb{R}}w^{2}|\mathscr{S}_{\varphi}^{\Omega}f(\xi,w)|^{2}dwd\xi\right)^{\frac{1}{2}}\left(b^{2}\mathscr{C}_{\varphi}\int_{\mathbb{R}}x^{2}|f(x)|^{2}dx\right)^{\frac{1}{2}}\geq\frac{\mathscr{C}_{\varphi}}{2b}||f||^{2}.$$

Equivalently,

$$\left(\int_{\mathbb{R}}\int_{\mathbb{R}}w^{2}\left|\mathscr{S}_{\varphi}^{\Omega}f(\xi,w)\right|^{2}dwd\xi\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}x^{2}\left|f(x)\right|^{2}dx\right)^{\frac{1}{2}}\geq\frac{\sqrt{\mathscr{C}_{\varphi}}}{2b^{2}}\|f\|^{2}.$$
(3.9)

This completes the proof.□

**Remark 3.3.** It is pertinent to mention that for the choices  $\Omega = (a/2b, -1/b, c/2b, 0, 0)$ ,  $\Omega = (\cot a/2, -\csc \alpha, \cot a/2, 0, 0)$ ,  $\alpha \neq n\pi$  and  $\Omega = (0, 1, 0, 0, 0)$ , Theorem 3.2 gives the corresponding Heisenberg's inequality associated with linear canonical S-transform, fractional S-transform and the classical S-transform, respectively.

**Lemma 3.2.** Let  $\mathscr{S}^{\Omega}_{\varphi}f$  be the quadratic phase S-transform of the signal  $f \in L^{2}(\mathbb{R})$ , then

$$\mathscr{C}_{\varphi} \int_{\mathbb{R}} \ln|x| \|f(x)\|^2 dx = \frac{1}{b^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \ln|x| \|\mathscr{C}_{\Omega}^{-1} \{\mathscr{S}_{\varphi}^{\Omega} f(\xi, w)\}(x)\|^2 dx d\xi.$$
(3.10)

**Theorem 3.4.** (*QPFT logarithmic uncertainty principle* [27]) Let  $\mathscr{C}_{\Omega}[f]$  be the quadratic phase Fourier transform of any signal  $f \in S(\mathbb{R})$ , then we have the following inequality:

$$b \int_{\mathbb{R}} \ln|w| \mathscr{Q}_{\Omega}[f](w)|^{2} dw + \int_{\mathbb{R}} \ln|x|| f(x)|^{2} dx + \geq \left[\frac{\Gamma'(1/4)}{\Gamma(1/4)} - \ln\pi - \ln|b|\right] \int_{\mathbb{R}} |f(x)|^{2} dx \qquad (3.11)$$

Now, applying Parseval's formula for the QPFT, (3.11) yields

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$$b \int_{\mathbb{R}} \ln|w| \|\mathscr{C}_{\Omega}[f](w)|^{2} dw + \int_{\mathbb{R}} \ln|x| \|\mathscr{C}_{\Omega}^{-1} \{\mathscr{C}_{\Omega}[f]\}(x)|^{2} dx$$
  

$$\geq b \left[ \frac{\Gamma'(1/4)}{\Gamma(1/4)} - \ln\pi - \ln|b| \right] \int_{\mathbb{R}} |\mathscr{C}_{\Omega}[f](w)|^{2} dw \qquad (3.12)$$

**Theorem 3.5.** (*QPST* logarithmic uncertainty principle) Let  $\varphi$ ,  $f \in S(\mathbb{R})$ , then the quadractic phase S-transform satisfies the following logrithmic inequality:

$$\begin{aligned} \int_{\mathbb{R}} \int_{\mathbb{R}} \ln|w| \mathscr{S}^{\Omega}_{\varphi} f(\xi, w)|^2 dw d\xi + b \, \mathscr{C}_{\varphi} \int_{\mathbb{R}} \ln|x| ||f(x)|^2 dx \\ &\geq \frac{1}{b} \left[ \frac{\Gamma'(1/4)}{\Gamma(1/4)} - \ln\pi - \ln|b| \right] \, \mathscr{C}_{\varphi} ||f||^2 \end{aligned}$$
(3.13)

**Proof.** It is obvious that  $\mathscr{Q}_{\Omega}[f]$  and  $\mathscr{S}_{\varphi}^{\Omega}f$  both are in  $\mathbb{S}(\mathbb{R})$  therefore on replacing  $\mathscr{Q}_{\Omega}[f]$  by  $\mathscr{S}_{\varphi}^{\Omega}f$ , (3.12) yields

$$b\int_{\mathbb{R}}\ln|w||\mathscr{S}^{\Omega}_{\varphi}f(\xi,w)|^{2}dw + \int_{\mathbb{R}}\ln|x||\mathscr{C}^{-1}_{\Omega}\{\mathscr{S}^{\Omega}_{\varphi}f(\xi,w)\}(x)|^{2}dx$$
  
$$\geq b\left[\frac{\Gamma'(1/4)}{\Gamma(1/4)} - \ln\pi - \ln|b|\right]\int_{\mathbb{R}}|\mathscr{S}^{\Omega}_{\varphi}f(\xi,w)|^{2}dw \qquad (3.14)$$

Integrating both sides of (3.14) with respect to  $d\xi$ , we get

$$\begin{split} b \int_{\mathbb{R}} \int_{\mathbb{R}} \ln|w| \|\mathscr{S}^{\Omega}_{\varphi} f(\xi,w)|^2 dw d\xi + \int_{\mathbb{R}} \int_{\mathbb{R}} \ln|x| \|\mathscr{C}^{-1}_{\Omega} \{\mathscr{S}^{\Omega}_{\varphi} f(\xi,w)\}(x)|^2 dx d\xi \\ \geq b \left[ \frac{\Gamma'(1/4)}{\Gamma(1/4)} - \ln\pi - \ln|b| \right] \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \mathscr{S}^{\Omega}_{\varphi} f(\xi,w) \right|^2 dw d\xi \quad . \end{split}$$

$$(3.15)$$

Applying Lemma 3.2 to the LHS of (3.15), we obtain

$$b \int_{\mathbb{R}} \int_{\mathbb{R}} \ln|w| |\mathscr{S}_{\varphi}^{\Omega} f(\xi, w)|^{2} dw d\xi + b^{2} \mathscr{C}_{\varphi} \int_{\mathbb{R}} \ln|x|| f(x)|^{2} dx$$

$$\geq b \left[ \frac{\Gamma'(1/4)}{\Gamma(1/4)} - \ln\pi - \ln|b| \right] \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \mathscr{S}_{\varphi}^{\Omega} f(\xi, w) \right|^{2} dw d\xi \qquad .$$

$$(3.16)$$

By virtue of (2.8), (3.16) yields

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \ln|w| |\mathscr{S}_{\varphi}^{\Omega} f(\xi, w)|^{2} dw d\xi + b \mathscr{C}_{\varphi} \int_{\mathbb{R}} \ln|x| |f(x)|^{2} dx$$

$$\geq \frac{1}{b} \left[ \frac{\Gamma'(1/4)}{\Gamma(1/4)} - \ln\pi - \ln|b| \right] \mathscr{C}_{\varphi} ||f||^{2} \qquad .$$

$$(3.17)$$

Which completes the proof.  $\Box$ 

## 4. Conclusion

In the study, we have accomplished three major objectives: first, we have introduced the notion of quadratic phase S-transform (QPST). Second, we establish the fundamental properties of the proposed transform, including the parseval's formula, inversion formula, shift and modulation. Third, we investigate Heisenberg uncertainty principle and logarithmic uncertainty principle associated with the QPST. In our future works we shall study the engineering background of the proposed transform.

## Declarations

- Availability of data and materials: The data is provided on the request to the authors.
- Competing interests: The authors have no competing interests.

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#### **Declaration of Competing Interest**

The authors report no declaration of interest.

## Data availability

Data will be made available on request.

#### Supplementary material

Supplementary material associated with this article can be found, in the online version, at 10.1016/j.prime.2023.100162

#### References

- L. Durak, O. Arikan, Short-time fourier transform: two fundamental properties and an optimal implementation, IEEE Trans. Signal Process. 51 (2003) 1231–1242.
- [2] H. Ji-Qu, C. Le-Ping, Wigner-ville distribution and its application in identification of FMCW, J. Yantai Univ. Nat. Sci. Eng. Ed. (2009).
- [3] O. Rioul, M. Vetterli, Wavelets and signal processing, IEEE Signal Process. Mag. 8 (1991) 14–38.
- [4] R.G. Stockwell, A basis for efficient representation of the s-transform, Digit. Signal Process. 17 (2007) 371–393.
- [5] H.M. Ozaktas, O. Aytür, Fractional fourier domains, Signal Process. 46 (1995) 119–124.
- [6] S.K. Sunil, The fractional s-transform on spaces of type s, J. Math. Vol. (2013). Article ID 105848, 9 pages
- [7] T.Z. Xu, B.Z. Li, Linear Canonical Transforms and its Applications, Science Press, Beijing, China, 2013.
- [8] B. Mawardi, T. Syamsuddin, L. Chrisandi, A generalized s-transform in linear canonical transform, J. Phys. 1341 (2019) 062005.
- [9] M.Y. Bhat, A.H. Dar, Multiresolution analysis for linear canonical s transform, Adv. Operator Theory 6 (2021) 68.
- [10] S. Drabycz, R.G. Stockwell, J.R. Mitchell, Image texture characterization using the discrete orthonormal s-transform, J. Digit. Imaging 22 (2009) 696–708.
- [11] J. Du, M.W. Wong, H. Zhu, Continuous and discrete inversion formulas for the stockwell transform, Integral Transforms Spec. Funct. 18 (2007) 537–543.
- [12] U. Battisti, L. Riba, Window-dependent bases for efficient representations of the stockwell transform, Appl. Comput. Harmon. Anal. 40 (2016) 292–320.

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- [13] A. Moukadem, Z. Bouguila, D.O. Abdeslam, A. Dieterlen, A new optimized stockwell transform applied on synthetic and real non-stationary signals, Digit. Signal Process. 46 (2015) 226–238.
- [14] L. Riba, M.W. Wong, Continuous inversion formulas for multi-dimensional modified stockwell transforms, Integral Transforms Spec. Funct. 26 (2015) 9–19.
- [15] S. Xu, L. Feng, Y. Chai, Y. He, Analysis of a-stationary random signals in the linear canonical transform domain, Signal Process. 146 (2018) 126–132.
- [16] R.G. Stockwell, L. Mansinha, R.P. Lowe, Localization of the complex spectrum: the s transform, IEEE Trans. Signal Process. 44 (1996) 998–1001.
- [17] S. Ventosa, C. Simon, M. Schimmel, J.J. Danobeitia, A. M'anuel, The s-transform from a wavelet point of view, IEEE Trans. Signal Process. 56 (7) (2008) 2771–2780.
- [18] P. Boggiatto, C. Fernandez, A. Galbis, A group representation related to the stockwell transform, Indiana Univ. Math. J. 58 (5) (2009) 2277–2304.
- [19] S. Drabycz, R.G. Stockwell, J.R. Mitchell, Image texture characterization using the discrete orthonormal s-transform, J. Digit. Imaging 22 (6) (2009) 696–708.
- [20] M. Hutníková, A. Miśková, Continuous stockwell transform: coherent states and localization operators, J. Math. Phys. 56 (2015) 073504.
- [21] A. Moukadem, Z. Bouguila, D.O. Abdeslam, A. Dieterlen, A new optimized stockwell transform applied on synthetic and real non-stationary signals, Digit. Signal Process. 46 (2015) 226–238.
- [22] L. Riba, M. Wong, Continuous inversion formulas for multi-dimensional modified stockwell transforms, Integral Transforms Spec. Funct. 26 (1) (2015) 9–19.
- [23] L.P. Castro, L.T. Minh, N.M. Tuan, New convolutions for quadratic-phase fourier integral operators and their applications, Mediterr. J. Math. 15 (2018) 1–17.
- [24] L.P. Castro, M.R. Haque, M.M. Murshed, S. Saitoh, N.M. Tuan, Quadratic fourier transforms, Ann. Funct. Anal. AFA 5 (1) (2014) 10–23.
- [25] S. Saitoh, Theory of reproducing kernels: applications to approximate solutions of bounded linear operator functions on hilbert spaces, Amer. Math. Soc. Trans. Ser. 230 (2) (2010) 107–134.
- [26] M.Y. Bhat, A.H. Dar, Towards quaternion quadratic-phase Fourier transform, Math. Methods Appl. Sci. (2023), https://doi.org/10.1002/mma.9126.
- [27] M.Y. Bhat, A.H. Dar, D. Urynbassarova, A. Urynbassarova, Quadratic-phase wave packet transform, Optik - Int. J. Light Electron Opt. (2022), https://doi.org/ 10.1016/j.ijleo.2022.169120.
- [28] M.Y. Bhat, A.H. Dar, The 2-d hyper-complex gabor quadratic-phase fourier transform, J. Anal. (2022), https://doi.org/10.1007/s41478-022-00445-7.
- [29] M.Y. Bhat, A.H. Dar, Quadratic-phase scaled Wigner distribution: Convolution and Correlation,, SIVP. (2023), https://doi.org/10.1007/s11760-023-02495-1.
- [30] M.Y. Bhat, A.H. Dar, I. Nurhidayat, S. Pinealas, An interplay of Wigner-Ville distribution and 2D Hyper-complex quadratic-phase Fourier transform, Fractal Fract. (2023), https://doi.org/10.3390/fractalfract7020159.
- [31] M.Y. Bhat, A.H. Dar, I. Nurhidayat, S. Pinealas, Uncertainty principles for the twosided quaternion windowed quadratic-phase Fourier transform, Symmetry (2023), https://doi.org/10.3390/sym14122650.
- [32] Z. Wei, T. Ran, W. Yue, Linear canonical S transform, Chinese J. Electron. 20 (1) (2011) 63–66.
- [33] D. Wei, Y. Zhang, Y.M. Li, Linear canonical Stockwell transform: theory and applications, IEEE Trans. on Signal Processing (2022), https://doi.org/10.1109/ TSP.2022.3152402.
- [34] W. Deyun, Z. Yijie, Fractional stockwell transform: theory and applications, Digit. SignalProcess. 115 (2021), 103090.