ON FRAMELET KERNELS OF M-BAND WAVELET FRAMES

FIRDOUS AHMAD SHAH[1*] AND MOHAMMAD YOUNUS BHAT[2]

ABSTRACT. Framelets and their promising features in applications have attracted a great deal of interest and effort in recent years especially in signal denoising, image compression, and numerical algorithms. In this note, we introduce reproducing kernel Hilbert spaces and associated kernels that are explicitly constructed by the elements of M-band framelet system.

1. INTRODUCTION

It is well known that the standard orthogonal wavelets are not also suitable for the analysis of high-frequency signals with relatively narrow bandwidth. To overcome this shortcoming, M-band orthonormal wavelets were created as a direct generalization of the 2-band wavelets [12]. The motivation for a larger $M(M > 2)$ comes from the fact that, unlike the standard wavelet decomposition which results in a logarithmic frequency resolution, the M-band decomposition generates a mixture of logarithmic and linear frequency resolution and hence generates a more flexible tiling of the time-frequency plane than that resulting from 2-band wavelet. The other significant difference between 2-band wavelets and M-band wavelets in construction lies in the aspect that the wavelet vectors are not uniquely determined by the scaling vector and the orthonormal bases do not consist of dilated and shifted functions through a single wavelet, but consist of ones by using $M - 1$ wavelets (see [1, 4, 7, 11]). It is this point that brings more freedoms for optimal wavelet bases.

However, the tight wavelet frames are different from the orthonormal wavelets because of redundancy. By sacrificing orthonormality and allowing redundancy, the tight wavelet frames become much easier to construct than the orthonormal wavelets. A catalyst for this development is the unitary extension principle (UEP) introduced by Ron and Shen [10], which provides a general construction of tight wavelet frames for $L^2(\mathbb{R}^n)$ in the shift-invariant setting, and included the pyramidal decomposition and reconstruction filter bank algorithms. The resulting tight wavelet frames are based on a multiresolution analysis, and the generators are often called *mother framelets*. The theory of tight wavelet frames has been extensively studied and well developed over the recent years. To mention only a

few references on tight wavelet frames, the reader is referred to [2, 3, 5, 8] and many references therein. In the M-band setting, Han and Cheng [6] have provided the general construction of M-band tight wavelet frames on $\mathbb{R}$ by following the procedure of Daubechies *et al.*[3] and Petukhov [8] via extension principles. They have presented a systematic algorithm for constructing tight wavelet frames generated by a given refinable function with dilation factor $M \geq 2$.

This paper is organized as follows. In Section 2, we review some basic facts about M-band tight wavelet frames using extension principles. In Section 3, we prove the main result of this note regarding the construction of reproducing kernels associated with the M-band framelet systems.

2. Definitions and Preliminarary Results

We begin this section by reviewing some major concepts concerning M-band wavelet frames. In the rest of this paper,, we use $\mathbb{N}, \mathbb{N}_0, \mathbb{Z}$ and $\mathbb{R}$ to denote the sets of all natural numbers, non-negative integers, integers and real numbers, respectively. The symbol I_M denotes the identity matrix of size $M > 2$.

The Fourier transform of a function $f \in L^1(\mathbb{R})$ is defined as usual by:

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-i\xi x} dx, \ \xi \in \mathbb{R}$$

and its inverse is

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi) e^{i\xi x} d\xi, \ x \in \mathbb{R}.$$

For given $\Psi := \{\psi_1, \ldots, \psi_L\} \subset L^2(\mathbb{R})$, define the M-band wavelet system

$$X(\Psi) := \left\{ \psi_{j,k}^\ell : 1 \leq \ell \leq L; j, k \in \mathbb{Z} \right\} \tag{2.1}$$

where $\psi_{j,k}^\ell = M^{j/2} \psi^\ell(M^j \cdot - k)$. The wavelet system $X(\Psi)$ is called a *M-band wavelet frame*, or simply a *wavelet frame*, if there exist positive numbers $0 < A \leq B < \infty$ such that for all $f \in L^2(\mathbb{R})$.

$$A \|f\|_2^2 \leq \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \left| \langle f, \psi_{j,k}^\ell \rangle \right|^2 \leq B \|f\|_2^2. \tag{2.2}$$

The largest constant A and the smallest constant B satisfying (2.2) are called the *lower* and *upper wavelet frame bounds*, respectively. A wavelet frame is a *tight wavelet frame* if A and B are chosen so that $A = B$ and then the generators $\psi_1, \psi_2 \ldots, \psi_L$ are often referred as *M-band framelets*. Furthermore, the wavelet frame is called a *Parseval wavelet frame* if $A = B = 1$, and in this case, every function $f \in L^2(\mathbb{R})$ can be written as

$$f(x) = \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \langle f, \psi_{j,k}^{\ell} \rangle \, \psi_{j,k}^{\ell}(x). \tag{2.3}$$

The construction of framelet systems often starts with the construction of MRA, which is built on refinable functions. A function $\varphi \in L^2(\mathbb{R})$ is called M-refinable if it satisfies a refinement equation:

$$\varphi(x) = \sum_{k\in\mathbb{Z}} h_0[k]\, \varphi(Mx - k). \tag{2.4}$$

for some $h_0 \in l^2(\mathbb{Z})$. The Fourier transform of (2.4) yields

$$\hat{\varphi}(\xi) = m_0\left(\frac{\xi}{M}\right) \hat{\varphi}\left(\frac{\xi}{M}\right), \tag{2.5}$$

where

$$m_0(\xi) = \frac{1}{M} \sum_{k\in\mathbb{Z}} h_0[k] e^{ik\xi},$$

is a 2π-periodic measurable function in $L^\infty[-\pi, \pi]$ and is often called the *refinement symbol* of φ. We further assume that:

$$\lim_{\xi\to 0} \hat{\varphi}(\xi) = 1, \quad \text{and} \quad \sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi + 2k\pi)|^2 \in L^\infty[-\pi, \pi]. \tag{2.6}$$

For a compactly supported refinable function $\varphi \in L^2(\mathbb{R})$, let V_0 be the closed shift invariant space generated by $\{\varphi(.-k) : k \in \mathbb{Z}\}$ and $V_j = \{\varphi(M^j.) : \varphi \in V_0\}$, $j \in \mathbb{Z}$. It is known that when φ is compactly supported, then $\{V_j\}_{j\in\mathbb{Z}}$ forms a multiresolution analysis (see [3]). Recall that a multiresolution analysis is a family of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ that satisfies: (i) $V_j \subset V_{j+1}, j \in \mathbb{Z}$, (ii) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$ and (iii) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$ (see [1, 11]). Let $\Psi := \{\psi_1, \ldots, \psi_L\} \subset V_1$, then

$$\hat{\psi}_\ell(\xi) = m_\ell\left(\frac{\xi}{M}\right) \hat{\varphi}\left(\frac{\xi}{M}\right), \tag{2.7}$$

where

$$m_\ell(\xi) = \frac{1}{M} \sum_{k\in\mathbb{Z}} h_\ell[k] e^{ik\xi}, \ \ell = 1, \ldots, L$$

are the 2π-periodic measurable functions in $L^\infty[-\pi, \pi]$ and are called the *framelet symbols*. The so-called *unitary extension principle* (UEP) provides a sufficient condition on Ψ such that the resulting M-band system $X(\Psi)$ forms a tight frame of $L^2(\mathbb{R})$. Han and Cheng in [6] gave a complete characterization of the M-band

tight wavelet frames via the unitary extension principle. The following is the fundamental tool they gave to construct M-band tight wavelet frames.

Theorem 2.1. *Suppose that the refinable function φ and the framelet symbols $m_0, m_1, \ldots, m_L$ satisfy (2.4)-(2.7). Define $\psi_1, \ldots, \psi_L$ by (2.7). Let $\mathcal{M}(\xi) = \left\{ m_\ell \left(\xi + \frac{2\pi p}{M} \right) \right\}_{\ell,p=0}^{M-1}$ such that $\mathcal{M}(\xi)\mathcal{M}^*(\xi) = I_M$, for a.e $\xi \in \sigma(V_0) := \left\{ \xi \in [-\pi, \pi] : \sum_{k \in \mathbb{Z}} |\hat{\varphi}(\xi + 2k\pi)|^2 \neq 0 \right\}$, then M-band wavelet system $X(\Psi)$ forms a tight wavelet frame for $L^2(\mathbb{R})$ with frame bound 1.*

3. Framelet Kernels Associated with M-band Wavelet Frames

In this section, we build reproducing kernel Hilbert spaces and associated kernels that are explicitly constructed by the elements of M-band framelet system. These M-band framelets are chosen in some scale ranges and are used as the basis of the feature space. Thus, one can combine the advantages of multiscale framelet representation with the merit of kernel methods.

Let $j_{\min}$ and $j_{\max}$ be the minimum and maximal scale indices. Then, for this choice of scale j, we consider a family of functions

$$\mathcal{F} := \left\{ \psi_{j,k}^\ell : j_{\min} \leq j \leq j_{\max}, k \in \mathbb{Z}, 1 \leq \ell \leq L \right\}. \tag{3.1}$$

Clearly, the above restricted scale range system $\mathcal{F}$ belongs to the M-band wavelet system $X(\Psi)$ given by (2.1). Next, we define a set of functions $\mathcal{H}$ by

$$\mathcal{H} = \left\{ f : f = \sum_{\ell=1}^{L} \sum_{j=j_{\min}}^{j_{\max}} \sum_{k \in \mathbb{Z}} h_{j,k}^\ell \, \psi_{j,k}^\ell \right\}.$$

For any $f, g \in \mathcal{H}$, the scale product in $\mathcal{H}$ is defined by

$$\langle f, g \rangle_{\mathcal{H}} = \sum_{\ell=1}^{L} \sum_{j=j_{\min}}^{j_{\max}} \sum_{k \in \mathbb{Z}} \langle f, \psi_{j,k}^\ell \rangle \langle g, \tilde{\psi}_{j,k}^\ell \rangle, \tag{3.2}$$

where

$$\tilde{\mathcal{F}} := \left\{ \tilde{\psi}_{j,k}^\ell : j_{\min} \leq j \leq j_{\max}, k \in \mathbb{Z}, 1 \leq \ell \leq L \right\}. \tag{3.3}$$

is the dual M-band wavelet frame of the system (3.1).

Theorem 3.1. *Suppose the M-band wavelet system $X(\Psi)$ given by (2.1) is a normalized tight wavelet frame for $L^2(\mathbb{R})$ generated by the UEP associated with the compactly supported M-refinable function φ. Then, $\left(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}} \right)$ is a reproducing kernel Hilbert space and its reproducing kernel is*

$$K(x,y) = \sum_{\ell=1}^{L} \sum_{j=j_{\min}}^{j_{\max}} \sum_{k\in\mathbb{Z}} \psi_{j,k}^{\ell}(x)\, \psi_{j,k}^{\ell}(y). \tag{3.4}$$

Proof. We split the proof of the theorem in the following three steps:

Step 1. Since $\mathcal{H}$ is a closed subspace of the Hilbert space $L^2(\Omega)$, where Ω is a compact dense subset of $\mathbb{R}$. Hence, it follows that $\mathcal{H}$ is also a Hilbert space with the same scale product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ in $L^2(\Omega)$. Moreover, it is proved in [9] that if a function space is spanned by a finite set of functions, then the function set is a frame for that space. Therefore, by using this result and the fact that M-band framelets are compactly supported, it follows that the restricted scale range system $\mathcal{F}$ given by (3.1) has a finite number of functions and hence, constitutes a frame for the space $\mathcal{H}$. Similarly, the dual wavelet system given by (3.3) constitutes a frame for $\mathcal{H}$. Thus, for any $f \in \mathcal{H}$, we have

$$f = \sum_{\ell=1}^{L} \sum_{j=j_{\min}}^{j_{\max}} \sum_{k\in\mathbb{Z}} \langle f, \psi_{j,k}^{\ell}\rangle \tilde{\psi}_{j,k}^{\ell} = \sum_{\ell=1}^{L} \sum_{j=j_{\min}}^{j_{\max}} \sum_{k\in\mathbb{Z}} \langle f, \tilde{\psi}_{j,k}^{\ell}\rangle \psi_{j,k}^{\ell}, \tag{3.5}$$

and there exists positive numbers A and B satisfying

$$A\|f\|_2^2 \leq \sum_{\ell=1}^{L} \sum_{j=j_{\min}}^{j_{\max}} \sum_{k\in\mathbb{Z}} \left|\langle f, \psi_{j,k}^{\ell}\rangle\right|^2 \leq B\|f\|_2^2$$

or equivalently

$$\frac{1}{B}\|f\|_2^2 \leq \sum_{\ell=1}^{L} \sum_{j=j_{\min}}^{j_{\max}} \sum_{k\in\mathbb{Z}} \left|\left\langle f, \tilde{\psi}_{j,k}^{\ell}\right\rangle\right|^2 \leq \frac{1}{A}\|f\|_2^2. \tag{3.6}$$

Step 2. We now prove that the scale product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ defined by (3.2) is valid in $\mathcal{H}$. To do so, let $f, g, h \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{R}$, then we have the following:

(i) $\quad \langle f, g\rangle_{\mathcal{H}} = \sum_{\ell=1}^{L} \sum_{j=j_{\min}}^{j_{\max}} \sum_{k\in\mathbb{Z}} \left\langle f, \tilde{\psi}_{j,k}^{\ell}\right\rangle \left\langle g, \tilde{\psi}_{j,k}^{\ell}\right\rangle = \langle g, f\rangle_{\mathcal{H}}.$

(ii) $\langle \alpha f + \beta g, h\rangle_{\mathcal{H}} = \sum_{\ell=1}^{L} \sum_{j=j_{\min}}^{j_{\max}} \sum_{k\in\mathbb{Z}} \left\langle \alpha f + \beta g, \tilde{\psi}_{j,k}^{\ell}\right\rangle \left\langle h, \tilde{\psi}_{j,k}^{\ell}\right\rangle$

$$
= \sum_{\ell=1}^{L} \sum_{j=j_{\min}}^{j_{\max}} \sum_{k\in\mathbb{Z}} \Big[\alpha \left\langle f, \tilde{\psi}_{j,k}^{\ell} \right\rangle + \beta \left\langle g, \tilde{\psi}_{j,k}^{\ell} \right\rangle \Big] \left\langle h, \tilde{\psi}_{j,k}^{\ell} \right\rangle
$$

$$
= \alpha \sum_{\ell=1}^{L} \sum_{j=j_{\min}}^{j_{\max}} \sum_{k\in\mathbb{Z}} \left\langle f, \tilde{\psi}_{j,k}^{\ell} \right\rangle \left\langle h, \tilde{\psi}_{j,k}^{\ell} \right\rangle + \beta \sum_{\ell=1}^{L} \sum_{j=j_{\min}}^{j_{\max}} \sum_{k\in\mathbb{Z}} \left\langle g, \tilde{\psi}_{j,k}^{\ell} \right\rangle \left\langle h, \tilde{\psi}_{j,k}^{\ell} \right\rangle
$$

$$
= \alpha \left\langle f, h \right\rangle_{\mathcal{H}} + \beta \left\langle g, h \right\rangle_{\mathcal{H}}.
$$

(iii)   Since

$$
\langle f, f \rangle_{\mathcal{H}} = \sum_{\ell=1}^{L} \sum_{j=j_{\min}}^{j_{\max}} \sum_{k\in\mathbb{Z}} \left| \left\langle f, \tilde{\psi}_{j,k}^{\ell} \right\rangle \right|^2 \geq 0.
$$

Therefore, equation (3.6) can be written as

$$
\frac{1}{B} \|f\|_2^2 \leq \langle f, f \rangle_{\mathcal{H}} \leq \frac{1}{A} \|f\|_2^2,
$$

which implies that

$$
\langle f, f \rangle_{\mathcal{H}} = 0 \quad \text{if and only if} \quad f = 0.
$$

Thus, we have proven that $\big(\mathcal{H}, \langle \cdot, \cdot \rangle \big)$ is a Hilbert space.

Step 3. Finally, we shall prove that $\mathcal{H}$ is a reproducing kernel Hilbert space with reproducing kernel given by (3.4). Since the restricted scale system given by (3.1) is a finite set of bounded functions. This naturally proves that the associated kernel

$$
K(x, y) = \sum_{\ell=1}^{L} \sum_{j=j_{\min}}^{j_{\max}} \sum_{k\in\mathbb{Z}} \psi_{j,k}^{\ell}(x)\, \psi_{j,k}^{\ell}(y),
$$

is a well defined function of $\mathcal{H}$, which has pointwise convergence. Therefore, for any $f \in \mathcal{H}$, equation (3.5) implies that

$$\Big\langle f(y), K(y,x)\Big\rangle_{\mathcal{H}}$$

$$= \sum_{\ell=1}^{L} \sum_{j=j_{\min}}^{j_{\max}} \sum_{k\in\mathbb{Z}} \Big\langle f(y), \tilde{\psi}_{j,k}^{\ell}(y)\Big\rangle \Big\langle \sum_{r=1}^{L} \sum_{m=j_{\min}}^{j_{\max}} \sum_{n\in\mathbb{Z}} \psi_{m,n}^{r}(x)\,\psi_{m,n}^{r}(y), \tilde{\psi}_{j,k}^{\ell}(y)\Big\rangle$$

$$= \sum_{\ell=1}^{L} \sum_{j=j_{\min}}^{j_{\max}} \sum_{k\in\mathbb{Z}} \Big\langle f(y), \tilde{\psi}_{j,k}^{\ell}(y)\Big\rangle \Big\langle \sum_{r=1}^{L} \sum_{m=j_{\min}}^{j_{\max}} \sum_{n\in\mathbb{Z}} \Big\langle \tilde{\psi}_{j,k}^{\ell}(y), \psi_{m,n}^{r}(y)\Big\rangle \psi_{m,n}^{r}(x)\Big\rangle$$

$$= \sum_{\ell=1}^{L} \sum_{j=j_{\min}}^{j_{\max}} \sum_{k\in\mathbb{Z}} \Big\langle f(y), \tilde{\psi}_{j,k}^{\ell}(y)\Big\rangle \psi_{j,k}^{\ell}(x)$$

$$= f(x).$$

Hence, the framelet kernel in (3.4) is an admissible reproducing kernel. $\square$

Example 3.2. Let

$$m_0(\xi) = \frac{\left(1 + e^{-i\xi}\right)^4}{16}$$

be the *refinement mask* of the 2-band refinable function φ, which is a piecewise cubic polynomial of order 4 supported on $[0,4]$.

Define the periodic measurable functions $m_\ell(\xi), \ell = 1, 2, 3, 4$ as follows:

$$m_1(\xi) = -\frac{\left(1 - e^{-i\xi}\right)^4}{4}, \qquad\qquad m_2(\xi) = -\frac{\left(1 - e^{-i\xi}\right)^3}{4}\left(1 + e^{-i\xi}\right),$$

$$m_3(\xi) = -\sqrt{6}\,\frac{\left(1 - e^{-i\xi}\right)^2}{16}\left(1 + e^{-i\xi}\right)^2, \quad m_4(\xi) = -\frac{\left(1 - e^{-i\xi}\right)}{4}\left(1 + e^{-i\xi}\right)^3,$$

Then, clearly $\{\psi_1, \psi_2, \psi_3, \psi_4\}$ generates a framelet system with the framelet kernel

$$k(x,y) = \sum_{\ell=1}^{4} \sum_{j=j_{\min}}^{j_{\max}} \sum_{k\in\mathbb{Z}} \psi_{j,k}^{\ell}(x)\,\psi_{j,k}^{\ell}(y).$$

References

1. N. Bi, X. Dai and Q. Sun, *Construction of compactly supported M-band wavelets*, Appl. Comput. Harmon. Anal. **6** (1999) 113-131.
2. A. Chai and Z. Shen, *Deconvolution: a wavelet frame approach*, Numer. Math. **106** (2007) 529-587.
3. I. Daubechies, B. Han, A. Ron and Z. Shen, *Framelets: MRA-based constructions of wavelet frames*, Appl. Comput. Harmon. Anal. **14** (2003) 1-46.

4. L. Debnath and F. A. Shah, *Wavelet Transforms and Their Applications*, Birkhäuser, Boston, 2015.

5. B. Han, *Wavelets and framelets within the framework of non-homogeneous wavelet systems*, In Approximation Theory XIII. Neamtu, M., Schumaker, L., Eds.; Springer, 2012., pp. 121-161.

6. J. C. Han and Z. Cheng, *The construction of M-band tight wavelet frames*, In Proceedings of the Third Intemational Conference on Machine Learning and Cybemetics. Shanghai. 2004, pp. 26-29.

7. T. Lin, S. Xu, Q. Shi and P. Hao, *An algebraic construction of orthonormal M-band wavelets with perfect reconstruction*, Appl. Comput. Harmon. Anal. **172** (2006) 717-730.

8. A. P. Petukhov, *Explicit construction of framelets*, Appl. Comput. Harmon. Anal. **11** (2001) 313-327.

9. A. Rakotomamonjy and S. Canu, *Frames, reproducing kernels, regularization and learning*, J. Mach. Learn. Res. **6** (2005) 1485-1515.

10. A. Ron and Z. Shen, *Affine systems in $L^2(\mathbb{R}^d)$: the analysis of the analysis operator*, J. Funct. Anal. **148** (1997) 408-447.

11. F. A. Shah and L. Debnath, *Explicit construction of M-band tight framelet packets*, Analysis. **32** (2012) 281-294.

12. P. Steffen, P. N. Heller, R. A. Gopinath and C. S. Burrus, *Theory of regular M-band wavelet bases*, IEEE Trans. Sig. Proces. **41** (12) (1993) 3497-3510.

[1] DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KASHMIR, SOUTH CAMPUS, ANANTNAG-192101, JAMMU AND KASHMIR, INDIA
E-mail address: fashah79@gmail.com

[2] DEPARTMENT OF MATHEMATICS, CENTRAL UNIVERSITY OF JAMMU, JAMMU-180011 , JAMMU AND KASHMIR, INDIA
E-mail address: gyounusg@gmail.com