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Quaternion Fractional Fourier Transform: Bridging Signal Processing and Probability Theory

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Abstract: The one-dimensional quaternion fractional Fourier transform (1DQFRFT) introduces a fractional-order parameter that extends traditional Fourier transform techniques, providing new insights into the analysis of quaternion-valued signals. This paper presents a rigorous theoretical foundation for the 1DQFRFT, examining essential properties such as linearity, the Plancherel theorem, conjugate symmetry, convolution, and a generalized Parseval's theorem that collectively demonstrate the transform's analytical power. We further explore the 1DQFRFT's unique applications to probabilistic methods, particularly for modeling and analyzing stochastic processes within a quaternionic framework. By bridging quaternionic theory with probability, our study opens avenues for advanced applications in signal processing, communications, and applied mathematics, potentially driving significant advancements in these fields.

Keywords: quaternion fractional Fourier transform; probability theory; quaternion algebra; characteristic function; stochastic processes; statistical analysis; quaternion-valued signals

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1. Introduction

The theory of fractional Fourier transforms (FRFTs) [1–9] has evolved significantly since its inception, primarily driven by the need to generalize the classical FT for various applications in optics, signal processing, and quantum mechanics. The FRFT, which interpolates between the identity operation and the FT, provides a powerful tool for analyzing signals in a rotated time–frequency plane. The quaternion fractional Fourier transform

(QFRFT) [10–12] further extends this concept into the quaternion domain, offering a more comprehensive framework for handling multi-dimensional and complex-valued signals.

The 1DQFRFT builds upon the foundations laid by the classical FRFT and the quaternion algebra, enabling the processing of signals with quaternionic components. This transform not only preserves the fundamental characteristics of the FRFT, such as linearity, shift, and modulation, but also introduces new capabilities specific to quaternionic signals. The 1DQFRFT has shown its versatility in various fields [13–20], including quantum mechanics, tensor-valued decomposition, color image processing, and three-dimensional signal analysis, where it effectively handles the challenges posed by multi-dimensional data. Quaternion algebras have also been instrumental in developing and solving various mathematical challenges. These include the formulation and resolution of quaternion linear systems [21], the analysis of linear differential equations within the quaternionic framework [22], and the advancement of sampling theories [23,24] that leverage quaternion properties. These applications demonstrate the algebras' versatility and power in addressing complex problems across multiple domains. Recent research has delve into one-dimensional signal processing [23–26].

In this article, we present a detailed introduction to the 1DQFRFT, highlighting its significance in probability theory. We explore key properties of the 1DQFRFT, such as convolution, in the context of probability density functions (PDFs). This exploration is crucial for understanding how quaternion-valued signals interact with systems or filters, particularly when dealing with complex structures like color images or signals in 3D space.

1.1. State of the Art: Quaternion-Valued Signals in Probability Theory

Quaternion algebra has gained considerable attention in probability theory due to its ability to effectively represent and manage multi-dimensional data. First introduced by W.R. Hamilton in 1843 [27], this mathematical framework provides a powerful tool for processing complex and multi-component signals. Its versatility has found applications in diverse fields, including 3D computer graphics, aerospace engineering, artificial intelligence, and color image processing.

Quaternion-valued signals enhance traditional methods by extending them to higher dimensions. Researchers have utilized quaternions to create sophisticated approaches for analyzing stochastic processes with multiple interconnected components. This work has led to the development of quaternion-valued random variables, probability density functions, and characteristic functions, offering fresh perspectives and tools for probabilistic analysis.

Significant advancements in this field include the development of Quaternion Fourier Transforms (QFTs) and their applications in signal processing. These transforms provide an efficient framework for managing quaternion-valued signals, maintaining their multi-dimensional structure while supporting operations like filtering and reconstruction. Researchers [26,28–33] have expanded the theoretical and practical understanding of quaternion signals within probability theory. Furthermore, quaternion-valued moment functions have been introduced and analyzed, offering tools to characterize the statistical properties of quaternion-valued random processes.

The integration of probability theory and quaternion algebra enriches theoretical frameworks while unlocking practical applications in fields that rely on multi-dimensional signal analysis. This interdisciplinary synergy fosters ongoing advancements, driving innovation and providing novel insights into the processing and interpretation of complex signals.

The QFT is a valuable tool but has the following limitations:

1. It is confined to frequency-domain analysis, lacking the flexibility of fractional-domain representations.
2. It does not readily connect with probabilistic tools like correlation or regression.

3. Its utility in analyzing multi-dimensional quaternion-valued signals is limited.

The 1DQFRFT addresses these gaps by the following:

1. Enabling fractional-domain analysis between time and frequency.
2. Establishing a novel bridge between signal processing and probability theory.
3. Providing tools for multi-dimensional quaternion signal analysis, expanding its scope to applications like physics and imaging.

This integration of probability theory and quaternion algebra enriches theoretical frameworks while addressing the limitations of the QFT. The 1DQFRFT not only enhances the flexibility of signal analysis through fractional-domain representations but also fosters an interdisciplinary synergy. This connection unlocks practical applications in fields that rely on multi-dimensional signal analysis, driving innovation and providing novel insights into the processing and interpretation of complex signals. By bridging the gap between signal processing and probability theory, the 1DQFRFT establishes itself as a transformative tool in quaternion-based analysis, opening new frontiers in applications ranging from physics to imaging and beyond.

This article is organized into distinct sections to systematically explore the integration of the QFRFT with probability theory. Section 2 delves into the 1DQFRFT, discussing its mathematical formulation and key properties. In Section 3, the focus shifts to applying the QFRFT within the context of probability theory, highlighting its role in analyzing quaternion-valued random variables and stochastic processes. Section 4 provides a discussion and analysis, critically evaluating the theoretical and practical implications of the proposed methods, supported by examples and comparisons. Finally, Section 5 concludes the article by summarizing the key findings and outlining potential avenues for future research.

1.2. Theoretical Foundations

We begin by reviewing the fundamental concepts and definitions related to quaternions. Quaternions are an extension of complex numbers and form an associative but non-commutative algebra over $\mathbb{R}$. The set of quaternions is denoted by $\mathbb{H}$. Any quaternion $q \in \mathbb{H}$ can be expressed in the following form [34]:

$$q = q_r + iq_i + jq_j + kq_k, \tag{1}$$

where $q_r, q_i, q_j, q_k \in \mathbb{R}$. Here, q_r represents the scalar part of q, denoted as $\mathrm{Sc}(q)$, while $iq_i + jq_j + kq_k$ represents the vector (or pure) part of q, conventionally denoted as $\vec{q}$.

The multiplication rules for the quaternion units are as follows:

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j, \tag{2}$$

$$i^2 = j^2 = k^2 = ijk = -1, \tag{3}$$

For two quaternions $p, q \in \mathbb{H}$ with vector parts $\vec{p}$ and $\vec{q}$, respectively, the quaternionic multiplication qp is given by

$$qp = q_r p_r - \vec{q} \cdot \vec{p} + q_r \vec{p} + \vec{q} p_r, \tag{4}$$

where

$$\vec{q} \cdot \vec{p} = q_i p_i + q_j p_j + q_k p_k \tag{5}$$

$$qp = i(q_j p_k - q_k p_j) + j(q_k p_i - q_i p_k) + k(q_i p_j - q_j p_i)$$

Analogous to the complex case, the quaternion conjugate of q is defined as follows:

$$\overline{q} = q_r - iq_i - jq_j - kq_k, \tag{6}$$

which is an anti-involution, meaning $\overline{qp} = \overline{p}\,\overline{q}$. Notice that the conjugate reverses the order of multiplication.

From (6), the norm or modulus of a quaternion $q \in \mathbb{H}$ is defined as follows:

$$|q| = \sqrt{q_r^2 + q_i^2 + q_j^2 + q_k^2}. \tag{7}$$

It follows that

$$\overline{q}q = q\overline{q} = |q|^2 \tag{8}$$

Using the conjugate (6) and modulus (7), the inverse of a non-zero quaternion $q \in \mathbb{H}$ is given by

$$q^{-1} = \frac{\overline{q}}{|q|^2}, \tag{9}$$

which shows that $\mathbb{H}$ is a normed division algebra. When $|q| = 1$, q is a unit quaternion. A quaternion q with $q_r = 0$ is called a pure quaternion, and its square is the negative of the sum of the squares of its components:

$$q^2 = -(q_i^2 + q_j^2 + q_k^2). \tag{10}$$

The scalar part of the product of two quaternions p and q can be obtained as follows:

$$\mathrm{Sc}(pq) = \frac{1}{2}(pq + qp) = p_r q_r - \vec{p} \cdot \vec{q} \tag{11}$$

A quaternion number q may also be represented as a complex number with complex and imaginary parts:

$$q = z_1 + jz_2, \tag{12}$$

where

$$z_1 = q_r + iq_i, \quad z_2 = q_j + iq_k.$$

This representation is known as the Cayley–Dickson form.

We define derivative operators for quaternions as follows:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x_r} + \frac{\partial}{\partial x_i} i + \frac{\partial}{\partial x_j} j + \frac{\partial}{\partial x_k} k \tag{13}$$

where $\frac{\partial}{\partial x_r}$ and similar terms denote partial derivatives with respect to the quaternion components.

Using the orthogonal planes split of $q \in \mathbb{H}$ with respect to the pure quaternion space, we define

$$q^+ = \frac{1}{2}(q + \overline{q}), \quad q^- = \frac{1}{2}(q - \overline{q}), \tag{14}$$

where q^+ is the symmetric part and q^- is the antisymmetric part of q.

In a similar manner to the complex case, an inner product for functions f and g in $\mathbb{H}$ is defined as follows:

$$(f,g)_{L^2(\mathbb{H};\mathbb{H})} = \int_{\mathbb{H}} f(x) \cdot g(x)\, d^4x,$$

where $x \in \mathbb{H}$ and $d^4x = dx_r dx_i dx_j dx_k$. Each quaternion function f can be decomposed as

$$f(x) = f_r(x) + f_i(x)i + f_j(x)j + f_k(x)k, \tag{15}$$

where $f_r(x), f_i(x), f_j(x)$, and $f_k(x)$ are real-valued functions of x.

In particular, for $f = g$, the $L^2(\mathbb{H};\mathbb{H})$ norm is defined as follows:

$$\|f\| = \left(\int_{\mathbb{H}} |f(x)|^2\, d^4x \right)^{1/2}. \tag{16}$$

2. One-Dimensional Quaternion Fractional Fourier Transform with Properties

The 1DQFRFT with angle β of a signal $f(t)$, denoted as $F_\beta(u)$, is defined as [35]

$$F_\beta(u) = A_\beta \int_{\mathbb{R}} f(t)\, K_\beta(u,t)\, dt, \tag{17}$$

where $K_\beta(u,t) = \exp\left(j\left(\frac{t^2+u^2}{2} \cot\beta - tu\csc\beta \right) \right)$ is the transforming kernel of the 1DQFRFT, and $A_\beta = \sqrt{\frac{1-j\cot(\beta)}{2\pi}}$ is the normalization constant.

The inverse transform is given by

$$f(t) = \int_{\mathbb{R}} F_\beta(u)\, K_\beta^*(u,t)\, du \tag{18}$$

Here, β represents the rotation angle and $*$ denotes complex conjugation. The term $A_\beta = \sqrt{\frac{1-j\cot(\beta)}{2\pi}}$ is a scaling factor, and F^β denotes the corresponding QFRFT with angle β. For $\beta = \frac{\pi}{2}$, the QFRFT simplifies to the QFT. From (17), we observe that if $f(t)$ is a real-valued function, we can interchange the position of the kernel $K_\beta(u,t)$ as follows:

$$\begin{aligned} F_\beta(u) &= \int_{\mathbb{R}} f(t) K_\beta(u,t)\, dt \\ &= \int_{\mathbb{R}} K_\beta(u,t) f(t)\, dt \end{aligned} \tag{19}$$

Theorem 1. *Let $f \in L^2(\mathbb{R};\mathbb{H})$; then, we have*

$$\int_{\mathbb{R}} |f(t)|^2\, dt = \int_{\mathbb{R}} |F_\beta(u)|^2\, du \tag{20}$$

This Plancherel theorem demonstrates the preservation of energy between the original signal $f(t)$ and its 1DQFRFT $F_\beta(u)$. This property is essential for analyzing quaternion-valued signals in the 1DQFRFT domain, similar to the classical Plancherel theorem for the FT. The theorem plays a vital role in signal processing applications by ensuring that key signal characteristics, such as energy or power, remain intact when transforming signals from the time domain to the frequency domain using the 1DQFRFT.

The 1DQFRFT preserves the L^2-norm of quaternion-valued functions, which implies that smoothness (in terms of continuity and differentiability) is preserved after the transformation. This is crucial in ensuring that the transformation does not distort the underlying signal in a way that would compromise its smoothness.

Theorem 2 (Conjugate Symmetry for 1DQFRFT)**.** *Let $f \in L^2(\mathbb{R};\mathbb{H})$ be a quaternion-valued function, and $F_\beta(u)$ be its 1DQFRFT with parameter β. Then, the following conjugate symmetry property holds:*

$$\overline{F_\beta(u)} = F_{-\beta}(u)$$

Proof. We begin by recalling the definition (17) of the 1DQFRFT $F_\beta(u)$:

$$F_\beta(u) = A_\beta \int_{\mathbb{R}} f(t)\, K_\beta(u,t)\, dt,$$

Now, let us consider the complex conjugate of $F_\beta(u)$:

$$\overline{F_\beta(u)} = \overline{A_\beta} \int_{\mathbb{R}} \overline{f(t)}\, \overline{K_\beta(u,t)}\, dt$$

Since the kernel $K_\beta(u,t)$ is a complex exponential, taking its conjugate gives

$$\overline{K_\beta(u,t)} = \exp\left(-j\left(\frac{t^2+u^2}{2}\cot\beta - tu\csc\beta\right)\right)$$

Notice that

$$\overline{K_\beta(u,t)} = K_{-\beta}(u,t)$$

Thus, we can rewrite the expression for $\overline{F_\beta(u)}$ as follows:

$$\overline{F_\beta(u)} = \overline{A_\beta} \int_{\mathbb{R}} \overline{f(t)}\, K_{-\beta}(u,t)\, dt$$

Since the normalization constant A_β depends on β, we have $\overline{A_\beta} = A_{-\beta}$. Therefore,

$$\overline{F_\beta(u)} = A_{-\beta} \int_{\mathbb{R}} \overline{f(t)}\, K_{-\beta}(u,t)\, dt$$

This is precisely the 1DQFRFT of $\overline{f(t)}$ with parameter $-\beta$:

$$\overline{F_\beta(u)} = F_{-\beta}(u)$$

Thus, the conjugate symmetry property is established.

$$\begin{aligned}
\overline{F_\beta(f)}(u) &= \int_{\mathbb{R}} A_\beta(f_r(t) - if_i(t) - jf_j(t) - kf_k(t))\exp\left(-j\frac{(t^2+u^2)}{2}\cot\beta - tu\csc\beta\right)dt \\
&= \int_{\mathbb{R}} A_\beta f_r(t)\left(\exp\left(-j\frac{(t^2+u^2)}{2}\cot\beta - tu\csc\beta\right)\right)dt \\
&\quad - i\int_{\mathbb{R}} A_\beta f_i(t)\left(\exp\left(-j\frac{(t^2+u^2)}{2}\cot\beta - tu\csc\beta\right)\right)dt \\
&\quad - j\int_{\mathbb{R}} A_\beta f_j(t)\left(\exp\left(-j\frac{(t^2+u^2)}{2}\cot\beta - tu\csc\beta\right)\right)dt \\
&\quad - k\int_{\mathbb{R}} A_\beta f_k(t)\left(\exp\left(-j\frac{(t^2+u^2)}{2}\cot\beta - tu\csc\beta\right)\right)dt \\
&= F_\beta(f_r)(u) - iF_\beta(f_i)(u) - jF_\beta(f_j)(u) - kF_\beta(f_k)(u).
\end{aligned}$$

which completes the proof. □

Theorem 3. *For $f \in L^p(\mathbb{R};\mathbb{H})$, where $p = 1,2$, the following holds:*

1. If f is a real signal or if $f(t) = f_r(t) + jf_j(t)$ (where $f_r(t)$ and $f_j(t)$ are the real and imaginary components, respectively), then

$$\overline{F_\beta(f)}(u) = F_{-\beta}(\overline{f})(u), \quad \forall u \in \mathbb{R}, \tag{21}$$

where $F_\beta(f)(u)$ is the 1DQFRFT of $f(t)$ with parameter β, and $\overline{f}$ denotes the quaternionic conjugate of f.

2. If f is a quaternionic signal, then it becomes

$$\overline{F_\beta(f)(u)} = F_{-\beta}(f_r)(u) - iF_\beta(f_i)(u) - jF_{-\beta}(f_j)(u) - kF_\beta(f_k)(u), \tag{22}$$

where $f(t) = f_r(t) + if_i(t) + jf_j(t) + kf_k(t)$ represents the components of the quaternionic signal f.

The proof of this theorem can be seen in [35]. This is known as the conjugate symmetry for the 1DQFRFT. One valuable tool associated with the 1DQFRFT is the convolution operator. We will now revisit the definition of convolution and its corresponding theorem for a brief overview.

Definition 1 (Quaternion Convolution for 1DQFRFT)**.** *Let $f, g \in L^1(\mathbb{R}; \mathbb{H})$. The convolution $(f * g)_\beta$ associated with the 1DQFRFT $F_\beta(u)$ of f and g is defined by*

$$(f * g)_\beta(t) = \int_{\mathbb{R}} f(y) g_\beta(t - y)\, dy, \tag{23}$$

where $g_\beta(t)$ is the 1DQFRFT of $g(t)$ with parameter β.

This definition extends the classical convolution operation to the quaternion-valued signals in the context of the 1DQFRFT.

Theorem 4 (Convolution Theorem)**.** *For $f \in L^2(\mathbb{R}; \mathbb{H})$ and $g \in L^1(\mathbb{R}; \mathbb{H})$, the following holds:*

$$F_\beta(f *_\beta g)(u) = F_\beta(f)(u) F_\beta(g)(u) e^{-i\beta u^2}, \quad \forall u \in \mathbb{R}.$$

Proof. For any $u \in \mathbb{R}$,

$$\begin{aligned}
F_\beta(f *_\beta g)(u) &= F_\beta\big(f_1 * g_1 - F_{-2\beta}(f_2) * g_2\big)(u) + jF_\beta\big(f_2 * g_1 + F_{-2\beta}(f_1) * g_1\big)(u) \\
&= F_\beta(f_1)(u) F_\beta(g_1)(u) e^{-i\beta u^2} - F_\beta\big(F_{-2\beta}(f_2)\big)(u) F_\beta(g_2)(u) e^{-i\beta u^2} \\
&\quad + j\Big[F_\beta(f_2)(u) F_\beta(g_1)(u) e^{-i\beta u^2} + F_\beta\big(F_{-2\beta}(f_1)\big)(u) F_\beta(g_1)(u) e^{-i\beta u^2}\Big] \\
&= \big[F_\beta(f_1)(u) F_\beta(g_1)(u) - F_\beta(F_{-2\beta}(f_2))(u) F_\beta(g_2)(u)\big] e^{-i\beta u^2} \\
&\quad + j\big[F_\beta(f_2)(u) F_\beta(g_1)(u) + F_\beta(F_{-2\beta}(f_1))(u) F_\beta(g_1)(u)\big] e^{-i\beta u^2} \\
&= \big[F_\beta(f_1)(u) F_\beta(g_1)(u) - F_{-\beta}(f_2)(u) F_\beta(g_2)(u)\big] e^{-i\beta u^2} \\
&\quad + j\big[F_\beta(f_2)(u) F_\beta(g_1)(u) + F_{-\beta}(f_1)(u) F_\beta(g_1)(u)\big] e^{-i\beta u^2} \\
&= F_\beta(f)(u) F_\beta(g)(u) e^{-i\beta u^2}.
\end{aligned}$$

Hence, the theorem is proved. □

The preservation of stability is shown by demonstrating that the 1DQFRFT of a convolution of two functions is equivalent to the product of their individual transforms. This ensures that the stability, in terms of boundedness and control over growth, is maintained under the transform.

Theorem 5 (Differential Property)**.** *Let $f(t) \in L^2(\mathbb{R};\mathbb{H})$. Then, the 1DQFRFT of the k-th derivative of $f(t)$ is given by*

$$F^{\beta}\left[\frac{d^k f(t)}{dt^k}\right](u) = \left(\cos\beta\frac{d}{du} + \sin\beta ju\right)^k F_{\beta}(u) \tag{24}$$

for all $k \geq 1$.

Proof. We prove this by mathematical induction on k. For $k = 1$, the property reduces to

$$F^{\beta}\left[f'(t)\right](u) = \left(\cos\beta\frac{d}{du} + \sin\beta ju\right)F_{\beta}(u). \tag{25}$$

This can be proven by differentiating the inverse 1DQFRFT with respect to t:

$$\begin{aligned} f(t) &= \int_{\mathbb{R}} F_{\beta}(u)\, K_{\beta}^{*}(u,t)\, du, \\ \frac{d}{dt}f(t) &= \int_{\mathbb{R}} F_{\beta}(u)\, \frac{\partial}{\partial t}K_{\beta}^{*}(u,t)\, du. \end{aligned}$$

The kernel $K_{\beta}(u,t)$ is given by

$$K_{\beta}(u,t) = \exp\left(j\left(\frac{t^2+u^2}{2}\cot\beta - tu\csc\beta\right)\right). \tag{26}$$

Differentiating with respect to t, we have

$$\begin{aligned} \frac{\partial}{\partial t}K_{\beta}^{*}(u,t) &= (-jt\cot\beta + ju\csc\beta)K_{\beta}^{*}(u,t) \\ &= j(u\csc\beta - t\cot\beta)K_{\beta}^{*}(u,t). \end{aligned} \tag{27}$$

Thus, we obtain

$$\frac{d}{dt}f(t) = \int_{\mathbb{R}} F_{\beta}(u)\, j(u\csc\beta - t\cot\beta)K_{\beta}^{*}(u,t)\, du. \tag{28}$$

Taking the 1DQFRFT of both sides and applying the property of the kernel function leads to the desired result in (25).

Assume that the property holds for $k = n$, i.e.,

$$F^{\beta}\left[\frac{d^n f(t)}{dt^n}\right](u) = \left(\cos\beta\frac{d}{du} + \sin\beta ju\right)^n F_{\beta}(u). \tag{29}$$

We need to prove that the property holds for $k = n+1$. Consider

$$\frac{d^{n+1}f(t)}{dt^{n+1}} = \frac{d}{dt}\left(\frac{d^n f(t)}{dt^n}\right).$$

Taking the 1DQFRFT of both sides,

$$\begin{aligned} F^{\beta}\left[\frac{d^{n+1}f(t)}{dt^{n+1}}\right](u) &= F^{\beta}\left[\frac{d}{dt}\left(\frac{d^n f(t)}{dt^n}\right)\right](u) \\ &= \left(\cos\beta\frac{d}{du} + \sin\beta ju\right)F^{\beta}\left[\frac{d^n f(t)}{dt^n}\right](u). \end{aligned}$$

Using the inductive hypothesis,

$$F^{\beta}\left[\frac{d^{n+1}f(t)}{dt^{n+1}}\right](u) = \left(\cos\beta\frac{d}{du} + \sin\beta ju\right)\left(\cos\beta\frac{d}{du} + \sin\beta ju\right)^{n} F_{\beta}(u) \tag{30}$$

$$= \left(\cos\beta\frac{d}{du} + \sin\beta ju\right)^{n+1} F_{\beta}(u). \tag{31}$$

Thus, the property holds for $k = n + 1$.

By mathematical induction, the property holds for all $k \geq 1$. □

The above differential property provides a framework for understanding how the 1DQFRFT interacts with signal derivatives. The operator involved ensures that smooth, differentiable signals retain their behavior after transformation, supporting the stability and smoothness of the signal.

Theorem 6 (Linearity)**.** *Given two quaternion-valued functions $f(t)$ and $g(t)$, and scalars a and b, we aim to prove the linearity of the 1DQFRFT:*

$$\mathcal{F}_{\beta}\{af(t) + bg(t)\}(u) = a\mathcal{F}_{\beta}\{f(t)\}(u) + b\mathcal{F}_{\beta}\{g(t)\}(u) \tag{32}$$

where the 1DQFRFT is defined as follows:

$$\mathcal{F}_{\beta}\{f(t)\}(u) = \int_{\mathbb{R}} f(t)K_{\beta}(u,t)\,dt$$

and $K_{\beta}(u,t)$ is the quaternionic kernel associated with the 1DQFRFT.

Proof. Let us consider

$$\begin{aligned}
\mathcal{F}_{\beta}\{af(t) + bg(t)\}(u) &= \int_{\mathbb{R}} [af(t) + bg(t)]K_{\beta}(u,t)\,dt \\
&= \int_{\mathbb{R}} \left[af(t)K_{\beta}(u,t) + bg(t)K_{\beta}(u,t)\right] dt \\
&= a\int_{\mathbb{R}} f(t)K_{\beta}(u,t)\,dt + b\int_{\mathbb{R}} g(t)K_{\beta}(u,t)\,dt \\
&= a\mathcal{F}_{\beta}\{f(t)\}(u) + b\mathcal{F}_{\beta}\{g(t)\}(u)
\end{aligned} \tag{33}$$

which shows that $F_{\beta}(u)$ is linear. □

3. One-Dimensional Quaternion Fractional Fourier Transform in Probability Theory

A quaternion-valued probability density function (PDF) $f_X(x)$ satisfies the following conditions:

1. **Positivity**: $f_a^X(x) \geq 0$, $f_b^X(x) \geq 0$, $f_c^X(x) \geq 0$, $f_d^X(x) \geq 0$ for all $x \in \mathbb{R}$.
2. **Normalization**:

   $$\int_{-\infty}^{\infty} f_X(x)\,dx = 1,$$

   where the integral is taken component-wise, i.e.,

   $$\int_{-\infty}^{\infty} f_a^X(x)\,dx + i\int_{-\infty}^{\infty} f_b^X(x)\,dx + j\int_{-\infty}^{\infty} f_c^X(x)\,dx + k\int_{-\infty}^{\infty} f_d^X(x)\,dx = 1.$$

When the 1DQFRFT is applied to $f_X(x)$, the resulting transform $\mathcal{F}_{\beta}(f_X)(u)$ captures the frequency domain representation of the quaternion-valued function in the context of the QFRFT parameter β. Mathematically, the 1DQFRFT of $f_X(x)$ is defined as follows:

$$\mathcal{F}_\beta(f_X)(u) = \int_{\mathbb{R}} f_X(x) K_\beta(x,u)\,dx,$$

where $K_\beta(x,u)$ is the kernel of the 1DQFRFT, which depends on the parameter β.

Expressing $f_X(x)$ in terms of its components, we can write

$$\mathcal{F}_\beta(f_X)(u) = \sum_{\alpha=a,b,c,d} e_\alpha \int_{\mathbb{R}} f_\alpha^X(x) K_\beta(x,u)\,dx,$$

where e_α represents the quaternionic units $1, i, j$, and k, respectively, and $f_\alpha^X(x)$ are the scalar components of $f_X(x)$.

Definition 2. *The quaternion cumulative distribution function (CDF) $F_X(x)$ for the 1DQFRFT $F_\beta(u)$ can be expressed as the derivative of the quaternion cumulative distribution function $F_X(x)$ with respect to x:*

$$f_X(x) = \frac{d}{dx} F_X(x), \tag{34}$$

where the probability P is related to $F_X(x)$ by

$$F_X(x) = P(X \leq x).$$

Applying the 1DQFRFT to the CDF

$$\mathcal{F}_\beta[F_X](u) = \int_{\mathbb{R}} F_X(x) K_\beta^j(x,u)\,dx, \tag{35}$$

where $K_\beta^j(x,u)$ is the kernel of the 1DQFRFT.

The corresponding PDF in the 1DQFRFT domain can be found by differentiating $\mathcal{F}_\beta[F_X](u)$ with respect to x:

$$\mathcal{F}_\beta[f_X](u) = \frac{d}{dx} \mathcal{F}_\beta[F_X](u).$$

Definition 3 (Expected value)**.** *In the 1DQFRFT domain, the expected value is given by*

$$\mathbb{E}[X] = \int_{\mathbb{R}} x F_{f_X}^\beta(u)\,du, \tag{36}$$

where $F_{f_X}^\beta(u)$ is the 1DQFRFT of the PDF $f_X(x)$.

In the 1DQFRFT domain, the expected value captures the distribution of X across fractional Fourier orders, providing a more generalized view of the frequency-domain behavior of the random variable.

Definition 4 (The mean)**.** *The mean of a quaternion-valued random variable X in the context of the 1DQFRFT is defined similarly to the traditional mean, but it incorporates the fractional Fourier domain characteristics. Given a quaternion-valued random variable X with a quaternion PDF $f_X(x)$, the mean in the 1DQFRFT domain is given by*

$$\mathbb{E}[X] = \int_{\mathbb{R}} x F_{f_X}^\beta(u)\,du,$$

where $F_{f_X}^\beta(u)$ represents the 1DQFRFT of the PDF $f_X(x)$.

Table 1 presents an overview of the 1DQFT, 1DQLCT, and 1DQFRFT, emphasizing their roles in probability theory.

Table 1. An Overview of 1DQFT, 1DQLCT, and 1DQFRFT in probability theory.

Feature	1DQFT	1DQLCT	1DQFRFT
Transformation	$\mathcal{F}[f(t)](u) = \int_{\mathbb{R}} f(t)e^{-jut}\,dt$	$\mathcal{Q}_A[f(t)](u) = \int_{\mathbb{R}} f(t)K_A^j(t,u)\,dt$	$\mathcal{F}_\beta[f(t)](u) = \int_{\mathbb{R}} f(t)K_\beta^j(t,u)\,dt$
CDF	$F_X(x) = P(X \leq x)$	$F_X(x) = P(X \leq x)$	$F_X(x) = P(X \leq x)$
PDF	$f_X(x) = \frac{d}{dx}F_X(x)$	$f_X(x) = \frac{d}{dx}F_X(x)$	$f_X(x) = \frac{d}{dx}F_X(x)$
Expected Value	$\mathbb{E}[X] = \int_{\mathbb{R}} x f_X(x)\,dx$	$\mathbb{E}[X] = \int_{\mathbb{R}} x f_X(x)\,dx$	$\mathbb{E}[X] = \int_{\mathbb{R}} x f_X(x)\,dx$
Variance	$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$	$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$	$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$
Characteristic Function	$\varphi_X(u) = \mathbb{E}[e^{juX}] = \mathcal{F}[f_X](u)$	$\varphi_X(u) = \mathcal{Q}_A[f_X](u)$	$\varphi_X(u) = \mathcal{F}_\beta[f_X](u)$
Differentiation Property	$\mathcal{F}\left\{\frac{d^n f(t)}{dt^n}\right\} = (-j)^n \mathcal{F}\{f(t)\}$	$\mathcal{Q}\left\{\frac{df(t)}{dt}\right\} = \left(-juc + a\frac{d}{du}\right)\mathcal{Q}\{f(t)\}$	$\mathcal{F}_\beta\left\{\frac{df(t)}{dt}\right\} = \left(\cos\beta\frac{d}{du} + \sin\beta ju\right)\mathcal{F}_\beta\{f(u)\}$
Convolution Theorem	$\mathcal{F}[f * g](u) = \mathcal{F}[f](u)\mathcal{F}[g](u)$	$\mathcal{Q}_A[f * g](u) = \mathcal{Q}_A[f](u)\mathcal{Q}_A[g](u)$	$\mathcal{F}_\beta[f * g](u) = \mathcal{F}_\beta[f](u)\mathcal{F}_\beta[g](u)$
Energy Preservation (Plancherel)	$\int_{\mathbb{R}} \lvert f(t)\rvert^2\,dt = \int_{\mathbb{R}} \lvert\mathcal{F}[f(u)]\rvert^2\,du$	$\int_{\mathbb{R}} \lvert f(t)\rvert^2\,dt = \int_{\mathbb{R}} \lvert\mathcal{Q}_A[f(u)]\rvert^2\,du$	$\int_{\mathbb{R}} \lvert f(t)\rvert^2\,dt = \int_{\mathbb{R}} \lvert\mathcal{F}_\beta[f(u)]\rvert^2\,du$

Example 1. *Consider a quaternion-valued random variable $X(t)$ with the quaternion PDF given by*

$$f_X(x) = f_a^X(x) + i f_b^X(x) + j f_c^X(x) + k f_d^X(x),$$

where $f_a^X(x), f_b^X(x), f_c^X(x)$, and $f_d^X(x)$ are the real-valued PDFs of the components X_a, X_b, X_c, and X_d, respectively.

Assume the following PDFs for the components:

$$f_a^X(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}, \quad f_b^X(x) = \frac{1}{2}e^{-|x|}, \quad f_c^X(x) = \frac{1}{\pi(1+x^2)}, \quad f_d^X(x) = \begin{cases} \frac{1}{2}e^{-x}, & x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

The mean of X in the 1DQFRFT domain is defined as follows:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x F_{f_X}^{\beta}(u)\,du,$$

where $F_{f_X}^{\beta}(u)$ is the 1DQFRFT of the PDF $f_X(x)$. The 1DQFRFT can be calculated component-wise.

For X_a (Gaussian Distribution):

$$F_{f_a^X}^{\beta}(u) = F^{\beta}\left\{\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\right\}(u)$$

The 1DQFRFT of a Gaussian function remains a Gaussian function (with possible scaling depending on β).

For X_b (Laplace Distribution):

$$F_{f_b^X}^{\beta}(u) = F^{\beta}\left\{\frac{1}{2}e^{-|x|}\right\}(u)$$

The 1DQFRFT of a Laplace distribution typically results in a function that resembles the original distribution but shifted and scaled in the fractional Fourier domain.

For X_c (Cauchy Distribution):

$$F_{f_c^X}^{\beta}(u) = F^{\beta}\left\{\frac{1}{\pi(1+x^2)}\right\}(u)$$

The Cauchy distribution in the 1DQFRFT domain will also produce a function related to the Cauchy distribution.

For X_d (Exponential Distribution):

$$F^{\beta}_{f_d^X}(u) = F^{\beta}\left\{\frac{1}{2}e^{-x} \text{ for } x \geq 0 \text{ and } 0 \text{ otherwise}\right\}(u)$$

The exponential distribution will transform into another function in the fractional Fourier domain.

The expected values can also be calculated component-wise.

For X_a:

$$\mathbb{E}[X_a] = \int_{-\infty}^{\infty} xF^{\beta}_{f_a^X}(u)\,du$$

Since the Gaussian function is symmetric around 0, the mean $\mathbb{E}[X_a]$ will be 0.

For X_b:

$$\mathbb{E}[X_b] = \int_{-\infty}^{\infty} xF^{\beta}_{f_b^X}(u)\,du$$

The mean of the Laplace distribution is 0, and this will remain the same in the 1DQFRFT domain (due to the symmetry and transformation properties).

For X_c:

$$\mathbb{E}[X_c] = \int_{-\infty}^{\infty} xF^{\beta}_{f_c^X}(u)\,du$$

The Cauchy distribution does not have a well-defined mean in the traditional sense due to its heavy tails, so this component contributes nothing to the mean.

For X_d:

$$\mathbb{E}[X_d] = \int_{-\infty}^{\infty} xF^{\beta}_{f_d^X}(u)\,du$$

The mean of the exponential distribution $f_d^X(x) = \frac{1}{2}e^{-x}$ is $1/2$, and this will remain the same in the fractional domain.

Combining all the results and substituting the calculated values, we obtain the mean of X in the 1DQFRFT domain as

$$\begin{aligned}\mathbb{E}[X] &= \mathbb{E}[X_a] + i\mathbb{E}[X_b] + j\mathbb{E}[X_c] + k\mathbb{E}[X_d] \\ &= 0 + i(0) + j(0) + k\left(\frac{1}{2}\right) \\ &= \frac{k}{2}.\end{aligned}$$

From above example, the following result can easily be verified

$$\overline{\mathbb{E}[X]} = \mathbb{E}[X_a] - i\mathbb{E}[X_b] - j\mathbb{E}[X_c] - k\mathbb{E}[X_d] \tag{37}$$

Example 2. *The quaternion-valued PDF is given by*

$$f_X(x) = \begin{cases} 1 + j \cdot 0.5 & \textit{for } x \in [0,1] \\ 0 & \textit{otherwise} \end{cases}$$

Using the definition (36), we compute the expected value of the real part as

$$\mathbb{E}[X_a] = \int_0^1 x\,dx = \frac{1}{2}$$

The expected value of the imaginary parts X_b and X_c are zero:

$$\mathbb{E}[X_b] = \mathbb{E}[X_c] = 0$$

The expected value of the imaginary part X_d is

$$\mathbb{E}[X_d] = \int_0^1 0.5 \cdot x\,dx = 0.25$$

Thus, the final expected value is

$$\mathbb{E}[X] = \frac{1}{2} + 0.25k$$

Definition 5. *Let X be a real random variable with the quaternion probability density function $f_X(x)$. The characteristic function of X, $\chi_X : \mathbb{R} \to \mathbb{H}$, is defined by the formula*

$$\chi_X(t) = \mathbb{E}[e^{itX}] = \int_{\mathbb{R}} f_X(x) e^{itx}\, dx. \tag{38}$$

The Equation (38) above may be expressed in the form

$$\begin{aligned}
\chi_X(t) &= \int_{\mathbb{R}} \Big(f_a^X(x) + i f_b^X(x) + j f_c^X(x) + k f_d^X(x) \Big) e^{itx}\, dx \\
&= \int_{\mathbb{R}} f_a^X(x) e^{itx}\, dx + i \int_{\mathbb{R}} f_b^X(x) e^{itx}\, dx + j \int_{\mathbb{R}} f_c^X(x) e^{itx}\, dx + k \int_{\mathbb{R}} f_d^X(x) e^{itx}\, dx \\
&= \chi_a^X(t) + i\chi_b^X(t) + j\chi_c^X(t) + k\chi_d^X(t),
\end{aligned} \tag{39}$$

where

$$\chi_i^X(t) = \int_{\mathbb{R}} f_i^X(x) e^{itx}\, dx, \quad i = a, b, c, d.$$

In the 1DQFRFT domain, the characteristic function is defined by applying the 1DQFRFT to the quaternion-valued probability density function:

$$\mathcal{F}_\beta\{\chi_X\}(u) = \int_{\mathbb{R}} \chi_X(t) K_\beta(t, u)\, dt.$$

The characteristic function $\mathcal{F}_\beta\{\chi_X\}(u)$ provides a comprehensive insight into the behavior of quaternion-valued distributions by capturing their transformation across fractional orders. This enables a nuanced analysis of characteristics such as mean and variance in fractional domains, making the 1DQFRFT particularly valuable for studying non-stationary signals and distributions with complex dynamics. The continuity of the characteristic function in the 1DQFRFT domain is guaranteed by the smoothness and boundedness of the kernel $K_\beta(t, u)$, along with the L^2-norm preservation ensured by the linearity and unitarity of the 1DQFRFT. In practical applications, high-resolution discretization and stable numerical methods play a crucial role in preserving these theoretical properties, maintaining smoothness and stability during fractional-order approximations.

Lemma 1. *Let $X(t)$ be a real random variable. Then,*

$$\mathcal{F}_\beta\{X(t)\}(u) = a\mathcal{F}_\beta\{X(t)\}(u) + ib\mathcal{F}_\beta\{X(t)\}(u) + jc\mathcal{F}_\beta\{X(t)\}(u) + kd\mathcal{F}_\beta\{X(t)\}(u). \tag{40}$$

Proof. In fact, we have

$$\mathcal{F}_\beta\{X(t)\}(u) = \int_{-\infty}^{\infty} X(t) K_\beta(t, u)\, dt,$$

from the definition (17) of the 1DQFRFT. This can be written as follows:

$$\mathcal{F}_\beta\{X(t)\}(u) = \int_{-\infty}^{\infty} (a f_X(x) + ib f_X(x) + jc f_X(x) + kd f_X(x)) K_\beta(t, u)\, dt.$$

The integral can be distributed:

$$\begin{aligned}
\mathcal{F}_\beta\{X(t)\}(u) = a &\int_{-\infty}^{\infty} f_X(x) K_\beta(t, u)\, dt \\
&+ ib \int_{-\infty}^{\infty} f_X(x) K_\beta(t, u)\, dt \\
&+ jc \int_{-\infty}^{\infty} f_X(x) K_\beta(t, u)\, dt \\
&+ kd \int_{-\infty}^{\infty} f_X(x) K_\beta(t, u)\, dt.
\end{aligned}$$

Therefore, we have

$$\mathcal{F}_\beta\{X(t)\}(u) = a\mathcal{F}_\beta\{X(t)\}(u) + ib\mathcal{F}_\beta\{X(t)\}(u) + jc\mathcal{F}_\beta\{X(t)\}(u) + kd\mathcal{F}_\beta\{X(t)\}(u).$$

This is the desired result. □

Lemma 2 (Riemann–Lebesgue Lemma)**.** *For a quaternion density function $f_X \in L^1(\mathbb{H};\mathbb{H})$, the characteristic function $\chi_X(t)$ satisfies the following:*

$$\lim_{|t|\to\infty} \chi_X(t) = 0.$$

Proof. Given that the characteristic function $\chi_X(t)$ of a random variable X is defined as

$$\chi_X(t) = \mathbb{E}[e^{itX}] = \int_{\mathbb{R}} f_X(x)e^{itx}\,dx,$$

and the 1DQFRFT with angle β is given (17)

$$F_\beta(u) = A_\beta \int_{\mathbb{R}} f(t)\,K_\beta(u,t)\,dt,$$

we will use the 1DQFRFT to demonstrate that $\chi_X(t) \to 0$ as $|t| \to \infty$.

The characteristic function $\chi_X(t)$ can be connected to the 1D QFRFT by examining how the QFRFT affects the density function f_X. The goal is to analyze the behavior of

$$\chi_X(t) = \int_{\mathbb{R}} f_X(x)e^{itx}\,dx.$$

This function behaves similarly to $F_\beta(u)$ when $\beta = \pi/2$ because the FRFT generalizes the FT.

The kernel $K_\beta(u,t)$ in the 1DQFRFT expression introduces a phase that oscillates rapidly when u is large:

$$K_\beta(u,t) = \exp\left(j\left(\frac{t^2+u^2}{2}\cot\beta - tu\csc\beta\right)\right).$$

For large $|u|$, the oscillations of $K_\beta(u,t)$ dominate the integral's behavior.

By substituting into the 1DQFRFT, we obtain

$$\chi_X(t) = \int_{\mathbb{R}} f_X(x)e^{itx}\,dx = \int_{\mathbb{R}} f_X(t)K_\beta(u,t)\,dt,$$

where we interpret $K_\beta(u,t)$ in terms of its oscillatory behavior when $\beta = \pi/2$ to approximate the Fourier-like behavior.

For $|t| \to \infty$, the integral

$$F_\beta(u) = A_\beta \int_{\mathbb{R}} f_X(t)\exp\left(j\left(\frac{t^2+u^2}{2}\cot\beta - tu\csc\beta\right)\right)dt$$

tends to zero because the highly oscillatory nature of $\exp(j(-tu\csc\beta))$ causes the terms to cancel out, especially when combined with the normalization factor A_β.

Thus,

$$\lim_{|t|\to\infty} \chi_X(t) = \lim_{|t|\to\infty} A_\beta \int_{\mathbb{R}} f_X(t)\exp\left(j\left(\frac{t^2+u^2}{2}\cot\beta - tu\csc\beta\right)\right)dt = 0,$$

due to the Riemann–Lebesgue Lemma for oscillatory integrals. □

The characteristic function $\chi_X(t)$, analyzed through the framework of the 1DQFRFT, shows that the oscillatory nature of the kernel as $|t| \to \infty$ ensures that $\chi_X(t) \to 0$. This conclusion aligns with the classical result but extends naturally to quaternion-valued functions and their fractional transforms, demonstrating the vanishing of the characteristic function at infinity.

Theorem 7 (Continuity)**.** *The characteristic function $\chi_X(t)$ of a quaternion-valued random variable X is continuous with respect to t.*

Proof. Using the definition of the characteristic function, we have

$$\chi_X(t) = \mathbb{E}[e^{itX}] = \int_{\mathbb{R}} f_X(x)e^{itx}\,dx,$$

where $f_X(x)$ is the probability density function (pdf) of X.

To show continuity, consider $|\chi_X(t+h) - \chi_X(t)|$:

$$|\chi_X(t+h) - \chi_X(t)| = \left| \int_{\mathbb{R}} f_X(x) \left(e^{i(t+h)x} - e^{itx} \right) dx \right|.$$

by factoring out e^{itx}:

$$|\chi_X(t+h) - \chi_X(t)| = \left| \int_{\mathbb{R}} f_X(x) e^{itx} \left(e^{ihx} - 1 \right) dx \right|.$$

Using the triangle inequality, we obtain

$$|\chi_X(t+h) - \chi_X(t)| \leq \int_{\mathbb{R}} |f_X(x)| \cdot |e^{ihx} - 1| dx.$$

Now, note that

$$|e^{ihx} - 1| \leq 2|f_X(x)|, \quad \text{and } 2|f_X(x)| \in L^1(\mathbb{R}),$$

ensuring that the integrand is absolutely integrable. By the Dominated Convergence Theorem,

$$\int_{\mathbb{R}} |f_X(x)| \cdot |e^{ihx} - 1| dx \to 0 \quad \text{as } h \to 0.$$

Thus, we conclude

$$\lim_{h \to 0} |\chi_X(t+h) - \chi_X(t)| = 0.$$

Therefore, $\chi_X(t)$ is continuous with respect to t. □

In the context of the 1DQFRFT, the continuity of the characteristic function implies that the transform preserves the smoothness and stability of the signal or probability distribution under transformations involving quaternionic variables. This continuity is essential for ensuring reliable signal analysis and processing when dealing with quaternion-valued data. The continuity property guarantees that small perturbations in the time or frequency domain result in proportionally small changes in the transformed space, which is crucial for practical applications in signal processing, communications, and control systems involving quaternion data representations.

Theorem 8 (Parseval's Identity for 1DQFRFT)**.** *Let $F_\beta(u)$ and $G_\beta(u)$ be the 1DQFRFT of functions $f_X(t)$ and $g_X(t)$, respectively, defined using the kernel $K_\beta(u,t)$. The 1DQFRFT is defined by*

$$F_\beta(u) = \int_{\mathbb{R}} f(t) K_\beta(u,t)\, dt,$$

where $K_\beta(u,t)$ is the kernel of the 1DQFRFT. Parseval's identity states

$$\int_{\mathbb{R}} G_\beta(t) F_\beta(t)^*\, dt = \int_{\mathbb{R}} \int_{\mathbb{R}} g_X(x) f_X(y) K_\beta(x,y)\, dx\, dy,$$

where $*$ denotes the quaternion conjugate.

Proof. Since $f_X(t)$, $g_X(t)$, and the 1DQFRFT kernel $K_\beta(u,t)$ are quaternions, we will decompose each into their scalar components:

$$f_X(t) = f_0(t) + f_1(t)i + f_2(t)j + f_3(t)k,$$

$$g_X(t) = g_0(t) + g_1(t)i + g_2(t)j + g_3(t)k,$$

$$K_\beta(u,t) = K_0(u,t) + K_1(u,t)i + K_2(u,t)j + K_3(u,t)k.$$

Each component function $f_n(t), g_n(t), K_n(u,t)$ (for $n = 0,1,2,3$) is real-valued.

The 1DQFRFT is computed component-wise:

$$F_{\beta,n}(u) = \int_{\mathbb{R}} f_n(t) K_n(u,t)\, dt,$$

$$G_{\beta,n}(u) = \int_{\mathbb{R}} g_n(t) K_n(u,t)\, dt.$$

The corresponding transformed components are multiplied component-wise. The inner product of the transformed functions is

$$\int_{\mathbb{R}} G_\beta(t) F_\beta(t)^* \, dt = \sum_{n=0}^{3} \int_{\mathbb{R}} G_{\beta,n}(t) F_{\beta,n}(t)\, dt.$$

Substituting the expressions for $F_{\beta,n}(t)$ and $G_{\beta,n}(t)$ yields

$$\int_{\mathbb{R}} G_{\beta,n}(t) F_{\beta,n}(t)\, dt = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} g_n(x) K_n(t,x)\, dx \right) \left(\int_{\mathbb{R}} f_n(y) K_n(t,y)\, dy \right) dt.$$

Applying Fubini's theorem component-wise to interchange the order of integration yields

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} K_n(t,x) K_n(t,y)\, dt\, g_n(x) f_n(y)\, dx\, dy.$$

Using the orthogonality property of the kernel components $K_n(u,t)$ yields

$$\int_{\mathbb{R}} K_n(t,x) K_n(t,y)\, dt = K_n(x,y).$$

Thus,

$$\int_{\mathbb{R}} G_{\beta,n}(t) F_{\beta,n}(t)\, dt = \int_{\mathbb{R}} \int_{\mathbb{R}} g_n(x) f_n(y) K_n(x,y)\, dx\, dy.$$

Summing the results over all components

$$\sum_{n=0}^{3} \int_{\mathbb{R}} G_{\beta,n}(t) F_{\beta,n}(t)\, dt = \sum_{n=0}^{3} \int_{\mathbb{R}} \int_{\mathbb{R}} g_n(x) f_n(y) K_n(x,y)\, dx\, dy,$$

yields

$$\int_{\mathbb{R}} G_\beta(t) F_\beta(t)^* \, dt = \int_{\mathbb{R}} \int_{\mathbb{R}} g_X(x) f_X(y) K_\beta(x,y)\, dx\, dy.$$

which completes the proof. □

Definition 6 (Variance). *For a real random variable X, the variance is defined as follows:*

$$\sigma^2 = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2. \tag{41}$$

The quaternionic characteristic function in the 1DQFRFT domain is given by

$$\mathcal{F}_\beta\{X(t)\}(u) = \int_{-\infty}^{\infty} f_X(x) K_\beta(t,u)\, dx, \tag{42}$$

where $K_\beta(t,u)$ is the kernel of the 1DQFRFT. To find the variance using the 1DQFRFT, we need the first and second derivatives of $\mathcal{F}_\beta\{X(t)\}(u)$ with respect to t at $t = 0$:

$$\frac{d}{dt} \mathcal{F}_\beta\{X(t)\}(u) \Big|_{t=0} \cdot (-i), \tag{43}$$

$$\frac{d^2}{dt^2} \mathcal{F}_\beta\{X(t)\}(u) \Big|_{t=0} \cdot (-i)^2. \tag{44}$$

The variance σ^2 can be computed using the following formula:

$$\sigma^2 = \frac{d^2}{dt^2} \mathcal{F}_\beta\{X(t)\}(u) \Big|_{t=0} \cdot (-i)^2 - \left(\frac{d}{dt} \mathcal{F}_\beta\{X(t)\}(u) \Big|_{t=0} \cdot (-i) \right)^2. \tag{45}$$

Example 3. *Consider a real random variable X uniformly distributed over the interval* $[-1, 1]$*. The probability density function (PDF) is as follows:*

$$f_X(x) = \begin{cases} \frac{1}{2} & \textit{if } -1 \leq x \leq 1, \\ 0 & \textit{otherwise}. \end{cases} \tag{46}$$

$\mathbb{E}[X]$*:*

$$\mathbb{E}[X] = \int_{-1}^{1} x \frac{1}{2} dx = 0.$$

$\mathbb{E}[X^2]$*:*

$$\mathbb{E}[X^2] = \int_{-1}^{1} x^2 \frac{1}{2} dx = \frac{1}{3}.$$

Variance σ^2*:*

$$\sigma^2 = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{1}{3}.$$

In the context of the 1DQFRFT, this variance can also be computed through the derivatives of the quaternionic characteristic function, as illustrated above. For this simple case, the result matches what you would get from the traditional method, but using 1DQFRFT provides a more generalized approach applicable to more complex quaternionic distributions.

4. Discussion and Analysis

The 1DQFRFT provides the most general framework, capable of handling both stationary and non-stationary processes, with the added benefit of fractional order, making it suitable for processes with fractional dynamics or other complex probabilistic structures. In summary, while the QFT provides the foundation, the QLCT and 1DQFRFT offer progressively more flexibility and capability in analyzing quaternionic signals and stochastic processes in probability theory. The choice of transform depends on the specific requirements of the problem at hand, such as the nature of the signal or process being studied and the desired level of detail in the analysis. While the 1DQFT expected value operates purely in the time domain, the 1DQLCT and 1DQFRFT expected values involve transformations that provide additional insights into how the quaternion random variable's distribution behaves under different transformations. The 1DQLCT focuses on shifts and scales, whereas the 1DQFRFT emphasizes fractional Fourier orders.

5. Conclusions and Future Perspectives

In this study, we investigated key probabilistic concepts within this framework, including the characteristic function, expected value, probability density function, and variance under the 1DQFRFT. These results highlight the potential of the 1DQFRFT to enhance the development of probability theory in the context of quaternion algebra, offering new avenues for research and application in fields requiring advanced signal processing techniques.

Future work will focus on further exploring the implications of the 1DQFRFT in uncertainty principles and how the quaternion probability density function interacts with its corresponding characteristic function. This exploration will deepen our understanding of the interplay between time–frequency localization and probabilistic properties in the quaternion setting.

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