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VECTOR-VALUED WAVELET PACKETS ON LOCAL FIELDS OF POSITIVE CHARACTERISTIC

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Abstract. The concept of vector-valued multiresolution analysis on local field of positive characteristic was considered by Abdullah [Vector-valued multiresolution analysis on local fields of positive characteristic, *Analysis.* 34(2014) 415-428]. We construct the associated wavelet packets for such an MRA and investigate their properties by virtue of the Fourier transform. Moreover, it is shown how to obtain several new bases of the space $L^2\left(K, \mathbb{C}^N\right)$ by constructing a series of subspaces of these vector-valued wavelet packets.

1. Introduction

In recent years there has been a considerable interest in the problem of constructing wavelet bases on various groups. R.L. Benedetto and J.J. Benedetto [**3**] developed a wavelet theory for local fields and related groups. They did not develop the multiresolution analysis (MRA) approach, their method is based on the theory of wavelet sets and only allows the construction of wavelet functions whose Fourier transforms are characteristic functions of some sets. Since local fields are essentially of two types: zero and positive characteristic (excluding the connected local fields $\mathbb{R}$ and $\mathbb{C}$). Examples of local fields of characteristic zero include the p-adic field $\mathbb{Q}_p$ where as local fields of positive characteristic are the Cantor dyadic group and the Vilenkin p-groups. Even though the structures and metrics of local fields of zero and positive characteristics are similar, but their wavelet and multiresolution analysis theory are quite different. The concept of multiresolution analysis on a local field K of positive characteristic was introduced by Jiang et al. [**8**]. They pointed out a method for constructing orthogonal wavelets on local field K with a constant generating sequence. Subsequently, tight wavelet frames on local fields of positive characteristic were constructed by Shah and Debnath [**11**] using extension principles. For more about wavelets and their applications, we refer the monograph [**7**].

It is well known that the classical orthonormal wavelet bases have poor frequency localization. For example, if the wavelet ψ is band limited, then the measure of the supp of $(\psi_{j,k})^\wedge$ is 2^j-times that of supp $\hat{\psi}$. To overcome this disadvantage, Coifman et al. [**6**] introduced the notion of orthogonal univariate wavelet packets. Well known Daubechies orthogonal wavelets are a special of wavelet packets. Chui and Li [**5**] generalized the concept of orthogonal wavelet packets to the case of non-orthogonal wavelet packets so that they can be employed to the spline wavelets and

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so on. Shen [**12**] generalized the notion of univariate orthogonal wavelet packets to the case of multivariate wavelet packets. The construction of wavelet packets and wavelet frame packets on local fields of positive characteristic were recently reported by Behera and Jahan in [**2**]. They proved lemma on the so-called splitting trick and several theorems concerning the Fourier transform of the wavelet packets and the construction of wavelet packets to show that their translates form an orthonormal basis of $L^2(K)$. Other notable generalizations are the vector-valued wavelet packets [**4**], wavelet packets and framelet packets related to the Walsh polynomials [**9**] and M-band framelet packets [**10**].

Recently, Abdullah [**1**] has generalized the classic theory of multiresolution analysis on Euclidean spaces $\mathbb{R}^n$ to vector-valued multiresolution analysis on local fields of positive characteristic. Motivated and inspired by the concept of vector-valued multiresolution analysis on local fields of positive characteristic, we construct the associated orthogonal wavelet packets for such an MRA on local fields of positive characteristic. More precisely, we show that the collection of all dilations and translations of the wavelet packets is an overcomplete system in $L^2\left(K, \mathbb{C}^N\right)$.

This paper is organized as follows. In Section 2, we discuss some preliminary facts about local fields of positive characteristic and introduce the notion of vector-valued multiresolution analysis on local field K. In Section 3, we construct vector-valued wavelet packets associated with vector scaling function Φ and show how they generate an orthonormal basis for $L^2\left(K, \mathbb{C}^N\right)$. In Section 4, we provide a direct decomposition for the space $L^2\left(K, \mathbb{C}^N\right)$ in terms of the vector-valued wavelet packets.

2. Preliminaries and Vector-valued MRA on Local Fields

Let K be a field and a topological space. Then K is called a *local field* if both K^+ and K^* are locally compact Abelian groups, where K^+ and K^* denote the additive and multiplicative groups of K, respectively. If K is any field and is endowed with the discrete topology, then K is a local field. Further, if K is connected, then K is either $\mathbb{R}$ or $\mathbb{C}$. If K is not connected, then it is totally disconnected. Hence by a local field, we mean a field K which is locally compact, non-discrete and totally disconnected. The p-adic fields are examples of local fields. More details are referred to [**13**]. In the rest of this paper, we use the symbols $\mathbb{N}, \mathbb{N}_0$ and $\mathbb{Z}$ to denote the sets of natural, non-negative integers and integers, respectively.

Let K be a local field. Let dx be the Haar measure on the locally compact Abelian group K^+. If $\alpha \in K$ and $\alpha \neq 0$, then $d(\alpha x)$ is also a Haar measure. Let $d(\alpha x) = |\alpha| dx$. We call $|\alpha|$ the *absolute value* of α. Moreover, the map $x \to |x|$ has the following properties: (a) $|x| = 0$ if and only if $x = 0$; (b) $|xy| = |x||y|$ for all $x, y \in K$; and (c) $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in K$. Property (c) is called the *ultrametric inequality*. The set $\mathfrak{D} = \{x \in K : |x| \leq 1\}$ is called the *ring of integers* in K. Define $\mathfrak{B} = \{x \in K : |x| < 1\}$. The set $\mathfrak{B}$ is called the *prime ideal* in K. The prime ideal in K is the unique maximal ideal in $\mathfrak{D}$ and hence as result $\mathfrak{B}$ is both principal and prime. Since the local field K is totally disconnected, so there exist an element of $\mathfrak{B}$ of maximal absolute value. Let $\mathfrak{p}$ be a fixed element of maximum absolute value in $\mathfrak{B}$. Such an element is called a *prime element* of K. Therefore, for such an ideal $\mathfrak{B}$ in $\mathfrak{D}$, we have $\mathfrak{B} = \langle \mathfrak{p} \rangle = \mathfrak{p}\mathfrak{D}$. As it was proved in [**13**], the set $\mathfrak{D}$ is compact and open. Hence, $\mathfrak{B}$ is compact and open. Therefore, the residue space $\mathfrak{D}/\mathfrak{B}$ is isomorphic to a finite field $GF(q)$, where $q = p^k$ for some prime p and $k \in \mathbb{N}$.

Let $\mathfrak{D}^* = \mathfrak{D} \setminus \mathfrak{B} = \{x \in K : |x| = 1\}$. Then, it can be proved that $\mathfrak{D}^*$ is a group of units in K^* and if $x \neq 0$, then we may write $x = \mathfrak{p}^k x', x' \in \mathfrak{D}^*$. For a proof of this fact we refer to [**17**]. Moreover, each $\mathfrak{B}^k = \mathfrak{p}^k\mathfrak{D} = \{x \in K : |x| < q^{-k}\}$ is a compact subgroup of K^+ and usually known as the *fractional ideals* of K^+. Let $\mathcal{U} = \{a_i\}_{i=0}^{q-1}$ be any fixed full set of coset representatives of $\mathfrak{B}$ in $\mathfrak{D}$, then every element $x \in K$ can be expressed uniquely as $x = \sum_{\ell=k}^{\infty} c_\ell \mathfrak{p}^\ell$ with $c_\ell \in \mathcal{U}$. Let χ be a fixed character on K^+ that is trivial on $\mathfrak{D}$ but is non-trivial on $\mathfrak{B}^{-1}$. Therefore, χ is constant on cosets of $\mathfrak{D}$ so if $y \in \mathfrak{B}^k$, then $\chi_y(x) = \chi(yx), x \in K$. Suppose that χ_u is any character on K^+, then clearly the restriction $\chi_u|\mathfrak{D}$ is also a character on $\mathfrak{D}$. Therefore, if $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representative of $\mathfrak{D}$ in K^+, then, as it was proved in [13], the set $\{\chi_{u(n)} : n \in \mathbb{N}_0\}$ of distinct characters on $\mathfrak{D}$ is a complete orthonormal system on $\mathfrak{D}$.

The Fourier transform $\hat{f}$ of a function $f \in L^1(K) \cap L^2(K)$ is defined by

$$\hat{f}(\xi) = \int_K f(x)\overline{\chi_\xi(x)}dx. \tag{2.1}$$

It is noted that

$$\hat{f}(\xi) = \int_K f(x)\,\overline{\chi_\xi(x)}dx = \int_K f(x)\chi(-\xi x)dx.$$

Furthermore, the properties of Fourier transform on local field K are much similar to those of on the real line. In particular Fourier transform is unitary on $L^2(K)$.

We now impose a natural order on the sequence $\{u(n)\}_{n=0}^{\infty}$. We have $\mathfrak{D}/\mathfrak{B} \cong GF(q)$ where $GF(q)$ is a c-dimensional vector space over the field $GF(p)$. We choose a set $\{1 = \zeta_0, \zeta_1, \zeta_2, \dots, \zeta_{c-1}\} \subset \mathfrak{D}^*$ such that span $\{\zeta_j\}_{j=0}^{c-1} \cong GF(q)$. For $n \in \mathbb{N}_0$ satisfying

$$0 \leq n < q,\ n = a_0 + a_1 p + \cdots + a_{c-1}p^{c-1},\ 0 \leq a_k < p,\ \text{and } k = 0, 1, \dots, c-1,$$

we define

$$u(n) = (a_0 + a_1\zeta_1 + \cdots + a_{c-1}\zeta_{c-1})\,\mathfrak{p}^{-1}. \tag{2.2}$$

Also, for $n = b_0 + b_1 q + b_2 q^2 + \cdots + b_s q^s,\ n \in \mathbb{N}_0,\ 0 \leq b_k < q, k = 0, 1, 2, \dots, s$, we set

$$u(n) = u(b_0) + u(b_1)\mathfrak{p}^{-1} + \cdots + u(b_s)\mathfrak{p}^{-s}. \tag{2.3}$$

This defines $u(n)$ for all $n \in \mathbb{N}_0$. In general, it is not true that $u(m+n) = u(m) + u(n)$. But, if $r, k \in \mathbb{N}_0$ and $0 \leq s < q^k$, then $u(rq^k + s) = u(r)\mathfrak{p}^{-k} + u(s)$. Further, it is also easy to verify that $u(n) = 0$ if and only if $n = 0$ and $\{u(\ell) + u(k) : k \in \mathbb{N}_0\} = \{u(k) : k \in \mathbb{N}_0\}$ for a fixed $\ell \in \mathbb{N}_0$. Hereafter we use the notation $\chi_n = \chi_{u(n)}$, $n \geq 0$.

Let the local field K be of characteristic $p > 0$ and $\zeta_0, \zeta_1, \zeta_2, \dots, \zeta_{c-1}$ be as above. We define a character χ on K as follows:

$$\chi(\zeta_\mu \mathfrak{p}^{-j}) = \begin{cases} \exp(2\pi i/p), & \mu = 0 \text{ and } j = 1, \\ 1, & \mu = 1, \dots, c-1 \text{ or } j \neq 1. \end{cases} \tag{2.4}$$

Next, we introduce the notion of vector-valued nonuniform multiresolution analysis on local field K of positive characteristic.

Let N be a constant and $2 \le N \in \mathbb{Z}$. By $L^2\big(K, \mathbb{C}^N\big)$, we denote the set of all vector-valued functions $\mathbf{f}(x)$ i.e.,

$$L^2\big(K, \mathbb{C}^N\big) = \Big\{\mathbf{f}(x) = \big(f_1(x), f_2(x), \ldots, f_N(x)\big)^T : x \in K, f_\ell(x) \in L^2(K), \ell = 1, 2, \ldots, N\Big\}$$

where T means the transpose of a vector. The space $L^2\big(K, \mathbb{C}^N\big)$ is called *vector-valued function space* on local field K of positive characteristic. For $\mathbf{f}(x) \in L^2\big(K, \mathbb{C}^N\big)$, $\|\mathbf{f}\|$ denotes the norm of vector-valued function $\mathbf{f}$ and is defined as:

$$\|\mathbf{f}\|_2 = \left(\sum_{\ell=1}^{N} \int_K \big|f_\ell(x)\big|^2 dx\right)^{1/2}. \tag{2.5}$$

For a vector-valued function $\mathbf{f}(x) \in L^2\big(K, \mathbb{C}^N\big)$, the integration of $\mathbf{f}(x)$ is defined as:

$$\int_K \mathbf{f}(x)dx = \left(\int_K f_1(x)dx, \int_K f_2(x)dx, \ldots, \int_K f_N(x)dx\right)^T.$$

Moreover, the Fourier transform of $\mathbf{f}(x)$ is defined by

$$\hat{\mathbf{f}}(\xi) = \int_K \mathbf{f}(x)\overline{\chi_\xi(x)}\, dx.$$

For any two vector-valued functions $\mathbf{f}, \mathbf{g} \in L^2\big(K, \mathbb{C}^N\big)$, their vector-valued inner product $\langle \mathbf{f}, \mathbf{g}\rangle$ is defined as:

$$\langle \mathbf{f}, \mathbf{g}\rangle = \int_K \mathbf{f}(x)\overline{\mathbf{g}(x)}\, dx. \tag{2.6}$$

Definition 2.1. Let K be a local field of positive characteristic. A *vector-valued multiresolution analysis* (VMRA) of $L^2\left(K, \mathbb{C}^N\right)$ is a sequence of closed subspaces $\{V_j : j \in \mathbb{Z}\}$ of $L^2\left(K, \mathbb{C}^N\right)$ satisfying:

(a) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;
(b) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2\left(K, \mathbb{C}^N\right)$;
(c) $\bigcap_{j\in\mathbb{Z}} V_j = \{\mathbf{0}\}$, where $\mathbf{0}$ is the zero vector of $L^2\left(K, \mathbb{C}^N\right)$;
(d) $\mathbf{f}(\cdot) \in V_j$ if and only if $\mathbf{f}(\mathfrak{p}^{-1}\cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$;
(e) there is a function $\mathbf{\Phi} \in V_0$, called the *scaling vector*, such that $\big\{\mathbf{\Phi}\big(x - u(k)\big) : k \in \mathbb{N}_0\big\}$ forms an orthonormal basis for V_0.

For $j \in \mathbb{Z}$, we define an MRA space $V_j \subset L^2\left(K, \mathbb{C}^N\right)$ as

$$V_j = \overline{\operatorname{span}}\left\{\mathbf{\Phi}\big(\mathfrak{p}^{-j}x - u(k)\big) : k \in \mathbb{N}_0\right\}, \quad j \in \mathbb{Z}.$$

Since $\mathbf{\Phi} = \big(\varphi_1, \varphi_2, \ldots, \varphi_N\big)^T \in V_0 \subset V_1$, there exists constant sequence $\{G_k : k \in \mathbb{N}_0\}$ such that

$$\mathbf{\Phi}(x) = \sqrt{q} \sum_{k\in\mathbb{N}_0} G_k\, \mathbf{\Phi}\big(\mathfrak{p}^{-1}x - u(k)\big). \tag{2.7}$$

Taking Fourier transform on both sides of (2.7), we obtain

$$\hat{\mathbf{\Phi}}(\xi) = H_0(\mathfrak{p}\xi)\, \hat{\mathbf{\Phi}}(\mathfrak{p}\xi), \tag{2.8}$$

where

$$H_0(\xi) = \frac{1}{\sqrt{q}} \sum_{k\in\mathbb{N}_0} G_k\, \overline{\chi_k(\xi)}. \tag{2.9}$$

Since χ_k is an integral periodic function on K, we have for any $s \in \mathbb{N}_0$

$$H_0\big(\xi + u(s)\big) = \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} G_k \overline{\chi_k\big(\xi + u(s)\big)} = \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} G_k \overline{\chi_k(\xi)} = H_0(\xi).$$

Hence, H_0 is also an integral periodic function on K. Let $W_j, j \in \mathbb{Z}$ be the orthogonal complement of V_j in V_{j+1}. Then, there exists $q-1$ vector-valued functions $\mathbf{\Psi}_\ell(x) \in L^2\left(K, \mathbb{C}^N\right)$, $\ell = 1, \ldots, q-1$, such that their translations and dilations form an orthonormal basis of W_j, i.e.,

$$W_j = \overline{\operatorname{span}}\left\{\mathbf{\Psi}_\ell\big(\mathfrak{p}^{-j}x - u(k)\big) : k \in \mathbb{N}_0, 1 \le \ell \le q-1\right\}, \quad j \in \mathbb{Z}. \tag{2.10}$$

Since $\mathbf{\Psi}_\ell \in W_0 \subset V_1$, there exists $q-1$ constant sequences $\left\{G_k^\ell : k \in \mathbb{N}_0\right\}$ such that

$$\mathbf{\Psi}_\ell(x) = \sqrt{q} \sum_{k \in \mathbb{N}_0} G_k^\ell\, \mathbf{\Phi}\big(\mathfrak{p}^{-1}x - u(k)\big). \tag{2.11}$$

By taking Fourier Transform, the refinement equation (2.11) becomes

$$\hat{\mathbf{\Psi}}_\ell(\xi) = H_\ell(\mathfrak{p}\xi)\, \hat{\mathbf{\Phi}}(\mathfrak{p}\xi), \tag{2.12}$$

where

$$H_\ell(\xi) = \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} G_k^\ell\, \overline{\chi_k(\xi)}. \tag{2.13}$$

We say $q-1$ vector-valued functions $\mathbf{\Psi}_1(x), \mathbf{\Psi}_2(x), \ldots, \mathbf{\Psi}_{q-1}(x)$, are orthogonal vector-valued wavelet functions associated with the orthogonal vector-valued scaling function $\mathbf{\Phi}(x)$ if they satisfy

$$\Big\langle \mathbf{\Psi}_\ell(x), \mathbf{\Phi}\big(x - u(k)\big)\Big\rangle = \delta_{0,k}\, \mathbf{I}_N, \quad k \in \mathbb{N}_0. \tag{2.14}$$

and the family $\left\{\mathbf{\Psi}_\ell(x), k \in \mathbb{N}_0, 1 \le \ell \le q-1\right\}$ is an orthonormal basis of the subspace W_0. Therefore, we have

$$\Big\langle \mathbf{\Psi}_\ell(x), \mathbf{\Psi}_{\ell'}\big(x - u(k)\big)\Big\rangle = \delta_{0,k}\delta_{\ell,\ell'}\, \mathbf{I}_N, \quad 1 \le \ell, \ell' \le q-1, k \in \mathbb{N}_0. \tag{2.15}$$

The following lemma, which will be used in next section, gives a characterization in the frequency domain of an orthogonal vector-valued function $\mathbf{f}(x)$.

Lemma 2.2. [**1**] *Let $\mathbf{f}(x) \in L^2\left(K, \mathbb{C}^N\right)$. Then $\mathbf{f}(x)$ is an orthogonal vector-valued function if and only if*

$$\sum_{k \in \mathbb{N}_0} \hat{\mathbf{f}}\big(\xi + u(k)\big)\, \hat{\mathbf{f}}^*\big(\xi + u(k)\big) = \mathbf{I}_N, \quad \xi \in K. \tag{2.16}$$

The vector-valued wavelets associated with the vector-valued multiresolution analysis $\{V_j : j \in \mathbb{Z}\}$ on local fields of positive characteristic has been recently characterized by Abdullah [**1**] in terms of the wavelet masks as:

Theorem 2.3. *Let $\mathbf{\Phi}(x) \in L^2\left(K, \mathbb{C}^N\right)$ be the orthogonal vector-valued scaling function of an VMRA $\{V_j : j \in \mathbb{Z}\}$. Then, $\mathbf{\Psi}_\ell(x) \in L^2\left(K, \mathbb{C}^N\right)$, $1 \le \ell \le q-1$ as defined by (2.11) are the associated orthogonal vector-valued wavelet functions if and only if*

$$\sum_{r=0}^{q-1} H_0\Big(\mathfrak{p}\big(\xi + u(r)\big)\Big) H_\ell\Big(\mathfrak{p}\big(\xi + u(r)\big)\Big)^* = \mathbf{0}, \quad 1 \le \ell \le q-1, \xi \in K, \tag{2.17}$$

and

$$\sum_{r=0}^{q-1} H_\ell\Big(\mathfrak{p}\big(\xi+u(r)\big)\Big) H_{\ell'}\Big(\mathfrak{p}\big(\xi+u(r)\big)\Big)^* = \delta_{\ell,\ell'}\,\mathbf{I}_N, \quad 1 \le \ell \le q-1, \xi \in K. \tag{2.18}$$

3. Vector-valued Wavelet Packets on Local Fields

In this Section, we construct vector-valued wavelet packets associated with vector-valued multiresolution analysis on local fields of positive characteristic. First of all, we have the following theorem.

Theorem 3.1. *Let $\mathbf{\Phi} \in L^2\left(K, \mathbb{C}^N\right)$ be such that $\big\{\mathbf{\Phi}\big(x-u(k)\big) : k \in \mathbb{N}_0\big\}$ is an orthonormal system in $L^2\left(K, \mathbb{C}^N\right)$ and let $V = \overline{span}\big\{\mathbf{\Phi}\big(\mathfrak{p}^{-1}x - u(k)\big)\big\}$. Let $\mathbf{\Psi}_\ell(x)$ and $H_\ell(\xi), 0 \le \ell \le q-1$, be the functions defined by (2.11) and (2.13), respectively, and satisfying the conditions (2.17) and (2.18). Then, $\big\{\mathbf{\Psi}_\ell\big(x-u(k)\big) : 0 \le \ell \le q-1,\, k \in \mathbb{N}_0\big\}$ is an orthonormal system. Also this system is an orthonormal basis for V if and only if it is orthonormal.*

Proof. Since $\mathbf{\Phi}$ is the orthogonal scaling vector associated with an VMRA. Therefore, by Lemma 2.2, we have

$$\sum_{k\in\mathbb{N}_0} \hat{\mathbf{\Phi}}\big(\xi+u(k)\big)\hat{\mathbf{\Phi}}^*\big(\xi+u(k)\big) = \mathbf{I}_N, \quad \xi \in K. \tag{3.1}$$

Also, for $0 \le \ell, \ell' \le q-1$ and $k \in \mathbb{N}_0$, we have

$$\begin{aligned}
&\Big\langle \mathbf{\Psi}_\ell(\cdot), \mathbf{\Psi}_{\ell'}\big(\cdot - u(k)\big)\Big\rangle \\
&= \sum_{r\in\mathbb{N}_0} \int_{r\mathfrak{D}} H_\ell(\mathfrak{p}\xi)\hat{\mathbf{\Phi}}(\mathfrak{p}\xi) H_{\ell'}(\mathfrak{p}\xi)^* \hat{\mathbf{\Phi}}(\mathfrak{p}\xi)^* \overline{\chi(k,\xi)}\, d\xi \\
&= \int_{\mathfrak{D}} \sum_{r\in\mathbb{N}_0} H_\ell\Big(\mathfrak{p}\big(\xi+u(r)\big)\Big)\hat{\mathbf{\Phi}}\Big(\mathfrak{p}\big(\xi+u(r)\big)\Big) H_{\ell'}\Big(\mathfrak{p}\big(\xi+u(r)\big)\Big)^* \hat{\mathbf{\Phi}}\Big(\mathfrak{p}\big(\xi+u(r)\big)\Big)^* \overline{\chi(k,\xi)}\, d\xi \\
&= \int_{\mathfrak{D}} \sum_{s=0}^{q-1} H_\ell\Big(\mathfrak{p}\big(\xi+u(s)\big)\Big) \left\{ \sum_{r\in\mathbb{N}_0} \hat{\mathbf{\Phi}}\Big(\mathfrak{p}\big(\xi+u(s)\big)+u(r)\Big)\hat{\mathbf{\Phi}}\Big(\mathfrak{p}\big(\xi+u(s)\big)+u(r)\Big)^* \right\} \\
&\qquad\qquad \times \quad H_{\ell'}\Big(\mathfrak{p}\big(\xi+u(s)\big)\Big)^* \overline{\chi(k,\xi)}\, d\xi
\end{aligned}$$

Using (2.17), (2.18) and (3.1), we obtain

$$\big\langle \mathbf{\Psi}_\ell(x), \mathbf{\Psi}_{\ell'}\big(x-u(k)\big)\big\rangle = \delta_{0,k}\,\delta_{\ell,\ell'}\,\mathbf{I}_N.$$

Hence, $\big\{\mathbf{\Psi}_\ell\big(x-u(k)\big) : 0 \le \ell \le q-1,\, k \in \mathbb{N}_0,\, x \in K\big\}$ is an orthonormal system.

For any function $\mathbf{f}(x) \in V$, there exists constant matrix sequences $\{C_k : k \in \mathbb{N}_0\}$ such that

$$\mathbf{f}(x) = \sqrt{q} \sum_{k\in\mathbb{N}_0} C_k\, \mathbf{\Phi}\big(\mathfrak{p}^{-1}x - u(k)\big). \tag{3.2}$$

Next, we claim that

$$\mathbf{\Phi}\big(\mathfrak{p}^{-1}x - u(k)\big) = q^2 \sum_{\ell=0}^{q-1} \sum_{r\in\mathbb{N}_0} \big(G^{\ell}_{k-qr}\big)^* \, \mathbf{\Phi}_\ell\big(x - u(r)\big). \tag{3.3}$$

Now, we have

$$\begin{aligned} q^2 \sum_{\ell=0}^{q-1} \sum_{r\in\mathbb{N}_0} & \big(G^{\ell}_{k-qr}\big)^* \, \mathbf{\Phi}_\ell\big(x - u(r)\big) \\ &= q^2 \sum_{\ell=0}^{q-1} \sum_{r\in\mathbb{N}_0} \big(G^{\ell}_{k-qr}\big)^* \sum_{s\in\mathbb{N}_0} \mathbf{\Phi}\big(\mathfrak{p}^{-1}x - \mathfrak{p}^{-1}u(r) - u(s)\big) \\ &= q^2 \sum_{s'\in\mathbb{N}_0} \left\{ \sum_{\ell=0}^{q-1} \sum_{r\in\mathbb{N}_0} \big(G^{\ell}_{k-qr}\big)^* \big(G^{\ell}_{s'-qr}\big) \right\} \mathbf{\Phi}\big(\mathfrak{p}^{-1}x - u(s')\big) \\ &= \mathbf{\Phi}\big(\mathfrak{p}^{-1}x - u(k)\big). \end{aligned}$$

Keeping in view of the equations (3.1) and (3.3), it follows that the system $\big\{\mathbf{\Psi}_\ell\big(x-u(k)\big) : 0 \le \ell \le q-1,\, k \in \mathbb{N}_0\big\}$ forms an orthonormal basis for V. □

For $n = 0, 1, \ldots,$ the *basic vector-valued wavelet packets* associated with the vector-valued multiresolution analysis on local fields of positive characteristic are defined recursively by

$$\Gamma_n(x) = \Gamma_{q\sigma+\ell}(x) = q^{1/2} \sum_{k\in\mathbb{N}_0} G^{\ell}_k \, \Gamma_\sigma\big(\mathfrak{p}^{-1}x - u(k)\big), \quad 0 \le \ell \le q-1, \tag{3.4}$$

where $\sigma \in \mathbb{N}_0$ is the unique element such that $n = q\sigma + \ell$, $0 \le \ell \le q-1$ holds. Implementation of Fourier transform to (3.4) yields

$$\left(\Gamma_{q\sigma+\ell}\right)^{\wedge}(\xi) = H_\ell(\mathfrak{p}\xi)\, \hat{\Gamma_\sigma}(\mathfrak{p}\xi), \quad 0 \le \ell \le q-1. \tag{3.5}$$

Theorem 3.2. *If $\{\Gamma_n(x), n \in \mathbb{N}_0\}$ are the vector-valued wavelet packets associated with VMRA $\{V_j : j \in \mathbb{Z}\}$ on local fields of positive characteristic. Then*

$$\big\langle \Gamma_n(x), \Gamma_n\big(x - u(k)\big)\big\rangle = \delta_{0,k}\, \mathbf{I}_N, \quad k \in \mathbb{N}_0. \tag{3.6}$$

Proof. We prove the theorem by induction on n. Since

$$\big\langle \Gamma_0(x), \Gamma_0\big(x - u(k)\big)\big\rangle = \big\langle \mathbf{\Phi}(x), \mathbf{\Phi}\big(x - u(k)\big)\big\rangle = \delta_{0,k}\, \mathbf{I}_N,$$

and, hence the claim is true for $n = 0$. Assume that (3.6) holds for $0 \le n \le t^\nu, \nu$ is a fixed integer. For $t^\nu \le n \le t^{\nu+1}$, we have $t^{\nu-1} \le [n/t] \le t^\nu$. Let $n = t[n/t] + \ell, 0 \le \ell \le q-1$. Then, by Lemma 2.2, we have

$$\Big\langle \Gamma_{[n/t]}(x), \Gamma_{[n/t]}\big(x - u(k)\big)\Big\rangle = \delta_{0,k}\mathbf{I}_N \Leftrightarrow \sum_{r\in\mathbb{N}_0} \Big\langle \hat{\Gamma}_{[n/t]}\big(\xi + u(r)\big), \hat{\Gamma}_{[n/t]}\big(\xi + u(r)\big)\Big\rangle = \mathbf{I}_N. \tag{3.7}$$

By (2.18), (3.5) and (3.7), we have

$$
\begin{aligned}
&\Big\langle \Gamma_n(x), \Gamma_n\big(x-u(k)\big)\Big\rangle \\
&\quad= \int_K \hat{\Gamma}_n(\xi)\,\hat{\Gamma}_n(\xi)^*\,\overline{\chi(k,\xi)}d\xi \\
&\quad= \int_{\mathfrak{D}} \sum_{r\in\mathbb{N}_0} H_\ell\Big(\mathfrak{p}\big(\xi+u(r)\big)\Big)\hat{\Gamma}_{q\sigma}\Big(\mathfrak{p}\big(\xi+u(r)\big)\Big)H^\ell\Big(\mathfrak{p}\big(\xi+u(r)\big)\Big)^* \\
&\qquad\qquad\times \hat{\Gamma}_{q\sigma}\Big(\mathfrak{p}\big(\xi+u(r)\big)\Big)^*\overline{\chi(k,\xi)}\,d\xi \\
&\quad= \int_{\mathfrak{D}} \sum_{s=0}^{q-1} H_\ell\Big(\mathfrak{p}\big(\xi+u(s)\big)\Big)\left\{\sum_{r\in\mathbb{N}_0}\hat{\Gamma}_{q\sigma}\Big(\mathfrak{p}\big(\xi+u(s)\big)+u(r)\Big)\hat{\Gamma}_{q\sigma}\Big(\mathfrak{p}\big(\xi+u(s)\big)+u(r)\Big)^*\right\} \\
&\qquad\qquad\times H_\ell\Big(\mathfrak{p}\big(\xi+u(s)\big)\Big)^*\overline{\chi(k,\xi)}\,d\xi \\
&\quad= \int_{\mathfrak{D}} \sum_{s=0}^{q-1} H^\ell\Big(\mathfrak{p}\big(\xi+u(s)\big)\Big)H_\ell\Big(\mathfrak{p}\big(\xi+u(s)\big)\Big)^*\overline{\chi(k,\xi)}\,d\xi \\
&\quad= \delta_{0,k}\,\mathbf{I}_N.
\end{aligned}
$$

□

Theorem 3.3. *Let $\{\Gamma_\gamma(x), \gamma\in\mathbb{N}_0\}$ be the vector-valued wavelet packets associated with VMRA $\{V_j : j\in\mathbb{Z}\}$ on local fields of positive characteristic. Then*

$$
\Big\langle \Gamma_{q\sigma+\ell}(x), \Gamma_{q\sigma+\ell'}\big(x-u(k)\big)\Big\rangle = \delta_{\ell,\ell'}\,\delta_{0,k}\,\mathbf{I}_N, \quad 0\le \ell,\ell'\le q-1,\ k\in\mathbb{N}_0. \tag{3.8}
$$

Proof. By (2.14), (2.18) and (3.7), we have

$$
\begin{aligned}
&\Big\langle \Gamma_{q\sigma+\ell}(x), \Gamma_{q\sigma+\ell'}\big(x-u(k)\big)\Big\rangle \\
&\quad= \int_K \hat{\Gamma}_{q\sigma+\ell}(\xi)\hat{\Gamma}_{q\sigma+\ell'}(\xi)^*\,\overline{\chi(k,\xi)}\,d\xi \\
&\quad= q\sum_{r\in\mathbb{N}_0}\int_{r\mathfrak{D}} H_\ell(\xi)\hat{\Gamma}_\sigma(\xi)H_{\ell'}(\xi)^*\hat{\Gamma}_\sigma(\xi)^*\,\overline{\chi\left(k,\mathfrak{p}^{-1}\xi\right)}\,d\xi \\
&\quad= q\int_{\mathfrak{D}} H_\ell(\xi)\left\{\sum_{r\in\mathbb{N}_0}\hat{\Gamma}_\sigma\big(\xi+u(r)\big)\hat{\Gamma}_\sigma\big(\xi+u(r)\big)^*\right\}H_{\ell'}(\xi)^*\,\overline{\chi\left(k,\mathfrak{p}^{-1}\xi\right)}\,d\xi \\
&\quad= q\int_{\mathfrak{p}\mathfrak{D}}\sum_{s=0}^{q-1} H_\ell\big(\xi+\mathfrak{p}u(s)\big)H_{\ell'}\big(\xi+\mathfrak{p}u(s)\big)^*\,\overline{\chi\left(k,\mathfrak{p}^{-1}\xi\right)}\,d\xi \\
&\quad= q\int_{\mathfrak{p}\mathfrak{D}} \delta_{\ell,\ell'}\mathbf{I}_N\,\overline{\chi\left(k,\mathfrak{p}^{-1}\xi\right)}\,d\xi \\
&\quad= \delta_{\ell,\ell'}\,\delta_{0,k}\,\mathbf{I}_N.
\end{aligned}
$$

This completes the proof. □

For the construction of vector-valued wavelet packets on local fields of positive characteristic, it is necessary to show that their translates form an orthonormal basis for $L^2\left(K,\mathbb{C}^N\right)$. This is evident from the following theorem.

Theorem 3.4. *Let $\{\Gamma_n(x), n\in\mathbb{N}_0\}$ be the basic vector-valued wavelet packets associated with VMRA $\{V_j : j\in\mathbb{Z}\}$ on local fields of positive characteristic. Then*

(i) $\left\{\Gamma_n\big(x-u(k)\big) : q^j\le n\le q^{j+1}-1,\, k\in\mathbb{N}_0\right\}$ *is an orthonormal basis of* W_j, $j\ge 0$.

(ii) $\left\{\Gamma_n\big(x-u(k)\big) : 0\le n\le q^j-1,\, k\in\mathbb{N}_0\right\}$ *is an orthonormal basis of* V_j, $j\ge 0$.

(iii) $\left\{\Gamma_n\big(x-u(k)\big) : n\ge 0,\, k\in\mathbb{N}_0\right\}$ *is an orthonormal basis of* $L^2\left(K,\mathbb{C}^N\right)$.

Proof. We use induction on j. Since $\{\Gamma_n : 1\le n\le q-1\}$ is the basic set of vector-valued wavelets, the case $j=0$ in (i) is trivial. Assume (i) holds for j. We shall prove it for $j+1$. By our assumption, the family

$$\left\{q^{1/2}\Gamma_n\big(\mathfrak{p}^{-1}x-u(k)\big) : q^j\le n\le q^{j+1}-1,\, k\in\mathbb{N}_0\right\}$$

is an orthonormal basis of W_{j+1}. Let

$$E_n=\overline{\operatorname{span}}\left\{q^{1/2}\Gamma_n\big(\mathfrak{p}^{-1}x-u(k)\big) : k\in\mathbb{N}_0\right\}.$$

Then, we have

$$W_{j+1}=\bigoplus_{n=q^j}^{q^{j+1}-1}E_n. \tag{3.9}$$

By applying Theorem 3.1 to E_n, we obtain functions $g_{\ell,n}, 0\le\ell\le q-1$, where

$$\hat{g}_{\ell,n}(\xi)=H_\ell(\mathfrak{p}\xi)\,\hat{\Gamma}_n(\mathfrak{p}\xi),\quad 0\le\ell\le q-1, \tag{3.10}$$

such that $\left\{g_{\ell,n}\big(\,\cdot-u(k)\big) : 0\le\ell\le q-1,\, k\in\mathbb{N}_0\right\}$ is an orthonormal basis of E_n. Using equation (3.5), we obtain $g_{\ell,n}=\Gamma_{qn+\ell}$. Since $\left\{\ell+qn : 0\le\ell\le q-1, q^j\le n\le q^{j+1}-1\right\}=\left\{n : q^{j+1}\le n\le q^{j+2}-1\right\}$. Hence, $\left\{\Gamma_n\big(x-u(k)\big) : q^{j+1}\le n\le q^{j+2}-1,\, k\in\mathbb{N}_0\right\}$ is an orthonormal basis of W_{j+1}. Thus we have proved (i) for $j+1$ and the induction is complete. Part (ii) follows from the fact that $V_j=V_0\oplus W_0\oplus\cdots\oplus W_{j-1}$, and Part (iii) from the following decomposition

$$L^2\left(K,\mathbb{C}^N\right)=V_0\oplus\left(\bigoplus_{j\ge 0}W_j\right).$$

□

4. The Direct Decomposition for Space $L^2\left(K,\mathbb{C}^N\right)$

In this section, we decompose the MRA-space V_j and wavelet space W_j by virtue of a series of subspaces of vector-valued wavelets packets on local fields. Furthermore, we present the direct decomposition for space $L^2\left(K,\mathbb{C}^N\right)$.
For $n=0,1,\dots$ and $j\in\mathbb{Z}$, we define

$$U_j^n=\overline{\operatorname{span}}\left\{q^{j/2}\Gamma_n\big(\mathfrak{p}^{-j}x-u(k)\big) : k\in\mathbb{N}_0\right\}. \tag{4.1}$$

Since $\Gamma_0=\varphi$ is the scaling function and $\{\Gamma_n : 1\le n\le q-1\}$ are the basic vector-valued wavelets, we observe that

$$U_j^0=V_j,\quad \bigoplus_{\ell=1}^{q-1}U_j^\ell=W_j,\quad j\in\mathbb{Z}.$$

Thus, the orthogonal decomposition $V_{j+1} = V_j \oplus W_j$ can be reformulated as

$$U_{j+1}^0 = \bigoplus_{\ell=0}^{q-1} U_j^\ell. \tag{4.2}$$

By virtue of (3.5), we can generalize the decomposition of U_{j+1}^n into q-orthogonal subspaces.

Proposition 4.1. *For $n \in \mathbb{N}_0$ and $j \in \mathbb{Z}$, we have the following formula*

$$U_{j+1}^n = \bigoplus_{\ell=0}^{q-1} U_j^{\ell+qn}. \tag{4.3}$$

Proof. From (4.1), we have

$$U_{j+1}^n = \overline{\text{span}}\Big\{q^{j+1/2}\Gamma_n\big(\mathfrak{p}^{-j-1}x - u(k)\big) : k \in \mathbb{N}_0\Big\}. \tag{4.4}$$

Let $\mathcal{P}_k(x) = \Big\{q^{j+1/2}\Gamma_n\big(\mathfrak{p}^{-j-1}x - u(k)\big) : k \in \mathbb{N}_0\Big\}$. Then $\mathcal{P}_k$ forms an orthonormal basis for the Hilbert space U_{j+1}^n. For $0 \le \ell \le q-1$, let

$$\mathcal{A}_\sigma^\ell(x) = \sum_{k\in\mathbb{N}_0} H_{k-q\sigma}^\ell \mathcal{P}_k(x), \quad \sigma \in \mathbb{N}_0, \tag{4.5}$$

and

$$\mathcal{S}_\ell = \overline{\text{span}}\left\{\mathcal{A}_\sigma^\ell : \sigma \in \mathbb{N}_0\right\}.$$

Then, by Theorem 3.1, we have

$$U_{j+1}^n = \bigoplus_{\ell=0}^{q-1} \mathcal{S}_\ell.$$

Therefore, equation (4.5) becomes

$$\begin{aligned}
\mathcal{A}_\sigma^\ell(x) &= \sum_{k\in\mathbb{N}_0} H_{k-q\sigma}^\ell \mathcal{P}_k(x) \\
&= \sum_{k\in\mathbb{N}_0} H_k^\ell\, \mathcal{P}_{k+q\sigma}(x) \\
&= q^{(j+1)/2} \sum_{k\in\mathbb{N}_0} H_k^\ell\, \Gamma_n\big(\mathfrak{p}^{-j-1}x - (u(k) + u(q\sigma))\big) \\
&= q^{(j+1)/2} q^{1/2} \sum_{k\in\mathbb{N}_0} H_k^\ell\, \Gamma_n\big(\mathfrak{p}^{-j}x - (u(qk) + u(\sigma))\big) \\
&= q^{j/2} \sum_{k\in\mathbb{N}_0} H_k^\ell\, \Gamma_n\big(\mathfrak{p}^{-j}x - (u(qk) + u(\sigma))\big) \\
&= \Gamma_{\ell+qn}\big(\mathfrak{p}^{-j}x - u(\sigma)\big).
\end{aligned}$$

Thus, we have

$$\mathcal{S}_\ell = \bigoplus_{\ell=0}^{q-1} U_{j+1}^n \quad \text{and} \quad U_{j+1}^n = \bigoplus_{\ell=0}^{q-1} U_j^{\ell+qn}.$$

This completes the proof. □

The above proposition can be used to obtain various decompositions of the wavelet subspaces $W_j, j \geq 0$.

Theorem 4.2. *For $j = 0, 1, \dots,$ we have*

$$W_j = \bigoplus_{\ell=1}^{q-1} U_j^\ell = \bigoplus_{\ell=q}^{q^2-1} U_{j-1}^\ell = \cdots = \bigoplus_{\ell=q^m}^{q^{m+1}-1} U_{j-m}^\ell = \cdots = \bigoplus_{\ell=q^j}^{q^{j+1}-1} U_0^\ell. \tag{4.6}$$

Proof. The proof is obtained by repeated application of the previous proposition. □

Theorem 3.2 can be used to construct various orthonormal bases of $L^2\left(K, \mathbb{C}^N\right)$. Let $S \subset \mathbb{N}_0 \times \mathbb{Z}$. We want to characterize the sets S such that the collection

$$\mathcal{F} = \left\{ q^{j/2}\Gamma_n\big(\mathfrak{p}^{-j}x - u(k)\big) : k \in \mathbb{N}_0,\, (n,j) \in S \right\}$$

will form an orthonormal basis of $L^2\left(K, \mathbb{C}^N\right)$. In other words, we are searching those subsets S of $\mathbb{N}_0 \times \mathbb{Z}$ for which

$$\bigoplus_{(n,j)\in S} U_j^n = L^2\left(K, \mathbb{C}^N\right). \tag{4.7}$$

Theorem 4.3. *Let $\{\Gamma_n : n \geq 0\}$ be the basic vector-valued wavelet packets associated with a VMRA $\{V_j : j \in \mathbb{Z}\}$ and $S \subset \mathbb{N}_0 \times \mathbb{Z}$. Then $\mathcal{F}$ is an orthonormal basis of $L^2\left(K, \mathbb{C}^N\right)$ if and only if $\{I_{n,j} : (n,j) \in S\}$ is a partition of $\mathbb{N}_0$, where*

$$I_{n,j} = \left\{ \ell \in \mathbb{N}_0 : q^j n \leq \ell \leq q^j(n+1) - 1 \right\}.$$

Proof. By the repeated application of Proposition 4.1, we have

$$\begin{aligned} U_j^n &= \bigoplus_{\ell=0}^{q-1} U_{j-1}^{\ell+qn} = \bigoplus_{\ell=qn}^{q(n+1)-1} U_{j-1}^\ell = \bigoplus_{\ell=qn}^{q(n+1)-1} \left[\bigoplus_{m=0}^{q-1} U_{j-2}^{m+q\ell} \right] \\ &= \bigoplus_{\ell=q^2 n}^{q^2(n+1)-1} U_{j-2}^\ell = \cdots = \bigoplus_{\ell=q^2 n}^{q^2(n+1)-1} U_0^\ell = \bigoplus_{\ell\in I_{n,j}} U_0^\ell. \end{aligned}$$

Therefore,

$$\bigoplus_{(n,j)\in S} U_j^n = \bigoplus_{(n,j)\in S} \bigoplus_{\ell\in I_{n,j}} U_0^\ell.$$

Using Theorem 3.4(iii), we get

$$L^2\left(K, \mathbb{C}^N\right) = \bigoplus_{\ell\in N_0} U_0^\ell.$$

Hence, (4.7) holds if and only if $\{I_{n,j} : (n,j) \in S\}$ is a partition of $\mathbb{N}_0$. □

References

[1] Abdullah, *Vector-valued multiresolution analysis on local fields of positive characteristic*, Analysis., **34** (2014), pp. 415-428.

[2] B. Behera and Q. Jahan, *Wavelet packets and wavelet frame packets on local fields of positive characteristic*, J. Math. Anal. Appl., **395** (2012), pp. 1-14.

[3] J.J. Benedetto and R.L. Benedetto, *A wavelet theory for local fields and related groups*, J. Geom. Anal., **14** (2004), pp. 423-456.

[4] Q.Chen and Z. Chang, *A study on compactly supported orthogonal vector-valued wavelets and wavelet packets*, Chaos, Solit. Fractals., **31** (2007), pp. 1024-1034.

[5] C.K. Chui and C. Li, *Non-orthogonal wavelet packets*, SIAM J. Math. Anal., **24** (1993), pp. 712-738.

[6] R.R. Coifman, Y. Meyer, S. Quake and M.V. Wickerhauser, *Signal processing and compression with wavelet packets*, Technical Report, Yale University, 1990.

[7] L. Debnath and F. Shah, *Wavelet Transforms and Their Applications*, Birkhäuser, New York, 2015.

[8] H.K. Jiang, D.F. Li and N. Jin, *Multiresolution analysis on local fields*, J. Math. Anal. Appl., **294** (2004), pp. 523-532.

[9] F.A. Shah and L. Debnath, *p-Wavelet frame packets on a half-line using the Walsh-Fourier transform*, Int. Transf. Spec. Funct., **22** (2011), pp. 907-917.

[10] F.A. Shah and L. Debnath, *Explicit construction of M-band tight framelet packets*, Analysis., **32** (2012), pp. 281-294.

[11] F.A. Shah and L. Debnath, *Tight wavelet frames on local fields*, Analysis., **33** (2013), pp. 293-307.

[12] Z. Shen, *Non-tensor product wavelet packets in $L^2(\mathbb{R}^s)$*, SIAM J. Math. Anal., **26** (1995), pp. 1061-1074.

[13] M.H. Taibleson, *Fourier Analysis on Local Fields*, Princeton University Press, Princeton, 1975.

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