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SOME KEY PROPERTIES OF THE GENERALISED TRIPLE HYPERGEOMETRIC FUNCTION OF SRIVASTAVA'S $H_{A,p,v}(w_1, w_2, w_3; w_4, w_5; z_1, z_2, z)$

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***Abstract*.** With the help of the generalised Beta function $B_{p,v}(x, y)$, we are able to create a generalised version of Srivastava's triple hypergeometric function $H_A(\cdot)$ associated with a numerical approaximation table in this paper, along with its integral expressions. Furthermore, we list some of its key properties, including the Mellin transform, a derivative identity, recurrence relations, and a bounded inequality . We also provide some integral expressions of this generalised $H_{A,p,v}(\cdot)$ function that use Meijers's G-function, the product of the Macdonald and Gauss hypergeometric functions. In addition, we compute a numerical approximation table of this generalised hypergeometric function $H_{A,p,v}(\cdot)$ with bounds by Wolfram Mathematica and computer algebraic software or objected oriented programme.

1. Introduction, definitions and preliminaries

Many areas of mathematical physics, statistics, economics, and other disciplines have a long history of using hypergeometric functions of a single variable. For the value of w_1, $w_2 \in \mathbf{C}$, $w_3 \in \mathbf{C}\backslash\mathbf{Z}_0^-$, the Gauss hypergeometric function is defined by [17]

$$ {}_2F_1\left(\begin{matrix} w_1, w_2 \\ w_3 \end{matrix}; z\right) = \sum_{n=0}^{\infty} \frac{(w_1)_n (w_2)_n}{(w_3)_n} \frac{z^n}{n!} \qquad (|z| < 1). \tag{1.1} $$

This hypergeometric function extensions includes w_j $(1 \leq j \leq p, q)$, which also has so many wide application; see [23].
The literature that is currently available on hypergeometric series includes this series and its generalisations in a number of application-related branches of mathematics.

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This kind of series shows up in quantum field theory quite naturally. Especially when calculating analytical expressions for Feynman integrals. On the other hand, the use of triple hypergeometric series known influences can really outcome in simplifications, problems being solved, or better understanding of quantum field theory. Srivastava and Karlsson described and analysed a table of numerous 205 triple hypergeometric functions in [22, Chapter 3]. Srivastava constructed a few complete triple hypergeometric functions of the second order, denoted as H_A, H_B and H_C, see cites [19, 20]. It is well known that Appell's hypergeometric functions F_1 and F_2 are generalised in H_C, H_B, respectively, while F_1 and F_2 are generalised in H_A. Also, Our work on Srivastava's triple hypergeometric function is motivated by the work given in [6, 7].
In this study, we focus on Srivastava's triple hypergeometric function H_A, which is written as [19], [22, p. 43] and [21, p.68]

$$H_A(w_1, w_2, w_3; w_4, w_5; z_1, z_2, z) = \sum_{i,j,k=0}^{\infty} \frac{(w_1)_{i+k}(w_2)_{i+j}(w_3)_{j+k}}{(w_4)_i(w_5)_{j+k}} \frac{z_1^i}{i!} \frac{z_2^j}{j!} \frac{z^k}{k!}, \tag{1.2}$$

$$= \sum_{i,j,k\geq 0} \frac{(w_1)_{i+j}\ (w_2)_{i+k}}{(w_4)_i} \frac{B(w_3+j+k, w_5-w_3)}{B(w_3, w_5-w_3)} \frac{z_1^i}{i!} \frac{z_2^j}{j!} \frac{z^k}{k!}. \tag{1.3}$$

To simplify the process, we could indeed add the parameter s to $H_A(\cdot)$ in the form:

$$H_A^{(\eta)}(w_1, w_2, w_3; w_4, w_5; z_1, z_2, z) := \sum_{i,j,k\geq 0} \left[\frac{(w_1)_{i+j}\ (w_2)_{i+k}}{(w_4)_i} \frac{B(w_3+\eta+j+k, w_5-w_3+\eta)}{B(w_3, w_5-w_3)} \frac{z_1^i}{i!} \frac{z_2^j}{j!} \frac{z^k}{k!} \right], \tag{1.4}$$

it diminish to (1.3) for $\eta = 0$. The region of convergence for $H_A(\cdot)$ function is $|z_1| < R_1,\ |z_2| < R_2$ and $|z| < R_3$, where R_1, R_2, R_3 satisfy the relation $R_1 + R_2 + R_3 = 1 + R_2 R_3$; see [10]. Here $(w)_\upsilon$ ($w, \upsilon \in \mathbb{C}$) is the symbol for the Pochhammer's rule (since $(1)_m = m!$), is usually defined by

$$(w)_\upsilon := \frac{\Gamma(w+\upsilon)}{\Gamma(w)} = \begin{cases} 1,\ (\upsilon = 0\ ;\ w \in \mathbb{C}\backslash\{0\}) \\ w(w+1)...(w+n-1), \quad (\upsilon = n \in \mathbb{N}\ ;\ w \in \mathbb{C}), \end{cases} \tag{1.5}$$

and $B(w, \upsilon)$ stands for the traditional Beta function as [13, (5.12.1)]

$$B(w, \upsilon) = \begin{cases} \displaystyle\int_0^1 t^{w-1}(1-t)^{\upsilon-1} dt, \quad (\Re(w) > 0, \Re(\upsilon) > 0), \\ \displaystyle\frac{\Gamma(w)\Gamma(\upsilon)}{\Gamma(w+\upsilon)}, \qquad (\Re(w) < 0, \Re(\upsilon) < 0), \quad (w, \upsilon) \in \mathbb{C} \setminus \mathbb{Z}_0^-). \end{cases} \tag{1.6}$$

The Mellin-Barnes contour integral is used to define the Meijer's G-function [21, p.45,eq.(1)]

$$G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} w_1,...,w_n; w_{n+1},...,w_p \\ v_1,...,v_m; v_{m+1},...,v_q \end{matrix}\right) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{\prod_{j=1}^{m}\Gamma(v_j-\mu)\prod_{j=1}^{n}\Gamma(1-w_j+\mu)}{\prod_{j=m+1}^{q}\Gamma(1-v_j+\mu)\prod_{j=n+1}^{p}\Gamma(w_j-\mu)} z^{\mu} d\mu\,, \tag{1.7}$$

where $z \neq 0$, and m, n, p, q are non negative integers such that $1 \leq m \leq q$; $0 \leq n \leq p$, and $p \leq q$. The function (1.7) converges in the region $|arg(z)| < \pi\kappa$ when $\kappa = m + n - \frac{1}{2}(p+q)$ and it is assumed that $\kappa > 0$. The specific cases of Meijer's G-function was derived by C. S. Meijer [8, 21].

$$G_{1,2}^{2,1}\left(2z \,\middle|\, \begin{matrix} \frac{1}{2} \\ a\ ,\ -a \end{matrix}\right) = \frac{e^{-z}\ K_a(z)}{\sqrt{\pi}}, \tag{1.8}$$

$$G_{1,2}^{2,1}\left(2z \,\middle|\, \begin{matrix} \frac{1}{2} \\ a\ ,\ -a \end{matrix}\right) = \frac{e^{-z}\ K_a(z)}{\sqrt{\pi}\ \cos(a\pi)}, \tag{1.9}$$

$$G_{0,2}^{2,0}\left(\frac{z^2}{4} \,\middle|\, \begin{matrix} \\ \frac{\mu+a}{2}\ ,\ \frac{\mu-a}{2} \end{matrix}\right) = 2\left(\frac{z}{2}\right)^{\mu} K_a(z), \tag{1.10}$$

$$G_{1,2}^{2,1}\left(2z \,\middle|\, \begin{matrix} \mu+\frac{1}{2} \\ \mu+a,\ \mu-a \end{matrix}\right) = \frac{e^{-z}\ K_a(z)}{(2z)^{-\mu}}, \tag{1.11}$$

$$G_{1,2}^{2,1}\left(2z \,\middle|\, \begin{matrix} \mu+\frac{1}{2} \\ \mu+a,\ \mu-a \end{matrix}\right) = \frac{\sqrt{\pi}\ e^{-z}\ K_a(z)}{(2z)^{-\mu}\ \cos(a\pi)}, \tag{1.12}$$

and

$$G_{0,4}^{4,0}\left(\frac{z^4}{256} \,\middle|\, \begin{matrix} \\ \frac{\mu+a}{4}\ ,\ \frac{2+\mu+a}{4},\ \frac{\mu-a}{4},\ \frac{2+\mu-a}{4} \end{matrix}\right) = 4\pi\left(\frac{z}{4}\right)^{\mu} K_a(z), \tag{1.13}$$

where μ is a free parameter and in all these expressions we have $z \neq 0$. The ${}_2F_1(\cdot)$ series is given an integral representation by [21, eq.(11)] and [23]

$${}_2F_1\left(\begin{matrix} w_1, w_2 \\ w_3 \end{matrix}; z\right) = \frac{\Gamma(w_3)}{\Gamma(w_2)\Gamma(w_3-w_2)}\int_0^1 \frac{t^{w_2-1}(1-t)^{w_3-w_2-1}}{(1-zt)^{w_1}} dt, \tag{1.14}$$

where $\Re(w_3) > \Re(w_2) > 0$ and $|\arg(1-z)| < \pi$.
In cites [15] and [12], an another form generalised Beta function is presented. A Beta function is given by Chaudhry et al. [2, p.20, Eq.(1.7)] and they demonstrated links between this generalisation and the Macdonald, erro r and Whittaker functions. Additionally, Chaudhry et al. [3] generalised the ${}_2F_1(\cdot)$ hypergeometric function and its integral form. The generalised Beta function $B(w_1; w_2; p)$ has recently been extended by Parmar et al. [16] by enhancing a parameter υ, which is described by

$$B_{p,\upsilon}(w_1, w_2) \equiv B(w_1, w_2; p, \upsilon) = \sqrt{\frac{2p}{\pi}}\int_0^1 t^{w_1-\frac{3}{2}}(1-t)^{w_2-\frac{3}{2}} K_{\upsilon+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right) dt, \tag{1.15}$$

where $\Re(p) > 0,\ \upsilon \geq 0$ and $K_{\upsilon+\frac{1}{2}}(\cdot)$ is the modified Bessel function of order $\upsilon+\frac{1}{2}$. If $\upsilon = 0$ in (1.15), then this function compress to $B(w_1, w_2; p)$, whereas $K_{\frac{1}{2}}(z) = (\pi/2z)^{\frac{1}{2}} e^{-z}$. Özarslan and Özergin [14] have expanded the double hypergeometric series $F_1(\cdot)$, which is given by [1]

$$F_1(w_1, w_2, w_3; w_4; z_1, z_2) := \sum_{j,i\geq 0} \frac{(w_2)_j (w_3)_i\ B(w_1+i+j, w_4-w_1)}{B(w_1, w_4-w_1)} \frac{z_1^i}{i!} \frac{z_2^j}{j!}, \quad (1.16)$$

where the region of covergence is $|z_1| < 1$ and $|z_2| < 1$. We recently obtained an extension of Appell's hypergeometric function $F_1(\cdot)$ and its integral expression, which we denoted by $F_{1,p,\upsilon}(\cdot)$. This work was published in [6].
The following is the paper's outline: `Section 2` defines the generalised Srivastava's triple hypergeometric function $H_{A,p,\upsilon}(\cdot)$, and `Section 3` presents integral expression for this function. `Sections (4-7)` illustrate the key characteristics of the $H_{A,p,\upsilon}(\cdot)$ function, including Mellin transforms, a differential formula, a bounded inequality, and recursion formulas. `Section 8` makes a few closing remarks.

2. Generalized version of Srivastava's triple hypergeometric function $H_{A,p,\upsilon}(\cdot)$

Srivastava invented the hypergeometric function $H_A(\cdot)$ and its integral expressions; for more information, see [18, 19, 21]. Based on the $B_{p,\upsilon}(w_1, w_2)$ function defined in (1.15), the succeeding (p, υ)-generalisation of $H_A(\cdot)$ function is being taken into consideration. Theorem provides this as a result.

Theorem 2.1. *Let the parameters $w_1, w_2, w_3 \in \mathbb{C}$ and $w_4, w_5 \in \mathbb{C}\backslash\mathbb{Z}_0^-$ holds true. Then*

$$H_{A,p,\upsilon}(w_1, w_2, w_3; w_4, w_5; z_1, z_2, z) = \sum_{i,j,k=0}^{\infty} \frac{(w_1)_{i+j}(w_2)_{i+k}}{(w_4)_i} \frac{B_{p,\upsilon}(w_3+j+k, w_5-w_3)}{B(w_3, w_5-w_3)} \frac{z_1^i}{i!} \frac{z_2^j}{j!} \frac{z^k}{k!}, \quad (2.1)$$

where $|z_1| < R_1,\ |z_2| < R_2$ and $|z| < R_3$, with $R_1 + R_2 + R_3 = 1 + R_2 R_3$ is the area of convergence. When $p = 0 = \upsilon$, the above expression clearly compresses into itself as a classical function.

Example 1. In Table-1 we compute numerical approximation values of the generalised Srivastava's triple hypergeometric function (2.1) to $H_{A,p,\upsilon}(\cdot)$ for different parameter values υ and p and set the range of indices upto 5th term for instance $i = j = k = 5$.

3. Some integral expression for $H_{A,p,\upsilon}(\cdot)$

Several authors investigate numerous integral expressions of the function $H_A(\cdot)$; for examples, see [4, 5, 9]. In this part, we express a number of definitive integral expressions of the $H_{A,p,\upsilon}(\cdot)$ function involving integrand as a product of algebraic functions, Macdonold functions and Gauss hypergeometric functions. Additionally, some integral expressions involving the Meijer's G-function are shown here.

TABLE 1. The numerical approximation table of a generalised function $H_{A,p,\upsilon}(\cdot)$ in (2.1) for distinct p and υ values when parameters $w_1 = 1/2$, $w_2 = 1/3$, $w_3 = 3/2$, $w_4 = 4$, $w_5 = 7/2$ and variables $z = 3$, $z_1 = 1$, $z_2 = 2$.

p	υ	$H_{A,p,\upsilon}(\cdot)$	p	υ	$H_{A,p,\upsilon}(\cdot)$
0.05	0.10	0.06191	0.35	0.40	5.19325
0.10	0.15	0.09414	0.40	0.45	13.2543
0.15	0.20	0.15064	0.45	0.50	31.4599
0.20	0.25	0.29346	0.50	0.55	69.2934
0.25	0.30	0.71193	0.55	0.60	141.978
0.30	0.35	1.92484	0.60	0.65	271.426

Theorem 3.1. *Let* $\upsilon \geq 0, \Re(w_j) > 0 (j = 1, 3, 4, 5), \Re(p) > 0,\ \Re(w_5) > \Re(w_3) > 0,\ |\arg(1 - z_2)| < \pi$ *and* $|\arg(1 - z)| < \pi$. *Then the following integral expressions holds true.*

$$\text{(i)}\ H_{A,p,\upsilon}(w_1, w_2, w_3; w_4, w_5; z_1, z_2, z) = \frac{\Gamma(w_5)}{\Gamma(w_3)\Gamma(w_5 - w_3)} \times \sqrt{\frac{2p}{\pi}} \int_0^1 t^{w_3-\frac{3}{2}}(1-t)^{w_5-w_3-\frac{3}{2}}(1-z_2t)^{-w_1}(1-zt)^{-w_2} \times K_{\upsilon+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right) {}_2F_1\left(\begin{matrix} w_1, w_2; \\ w_4; \end{matrix}\ Z_1\right) dt, \tag{3.1}$$

where

$$Z_1 = \frac{z_1}{(1 - z_2t)(1 - zt)},$$

$$\text{(ii)}\ H_{A,p,\upsilon}(w_1, w_2, w_3; w_4, w_5; z_1, z_2, z) = \frac{\Gamma(w_4)\Gamma(w_5)}{\Gamma(w_1)\Gamma(w_3)\Gamma(w_4 - w_1)\Gamma(w_5 - w_3)} \times \sqrt{\frac{2p}{\pi}} \int_0^1 \int_0^1 \phi_2\ K_{\upsilon+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right) d\xi dt, \tag{3.2}$$

where

$$\phi_2 = \xi^{w_1-1}t^{w_3-\frac{3}{2}}(1-\xi)^{w_4-w_1-1}(1-t)^{w_5-w_3-\frac{3}{2}}(1-z_2t)^{-w_1}(1-zt)^{-w_2},$$

$$\text{(iii)}\ H_{A,p,\upsilon}(w_1, w_2, w_3; w_4, w_5; z_1, z_2, z) = \frac{\Gamma(w_5)}{\Gamma(w_3)\Gamma(w_5 - w_3)} \times \sqrt{\frac{2p}{\pi}} \int_0^1 \xi^{w_3-\frac{3}{2}}(1+\xi)^{w_1+w_2-w_5+1}\{\Omega_1\}^{-w_1}\{\Omega_2\}^{-w_2} \times K_{\upsilon+\frac{1}{2}}\left(\frac{p(1+\xi)^2}{\xi}\right) {}_2F_1\left(\begin{matrix} w_1, w_2; \\ w_4; \end{matrix}\ z_1\Lambda\right) d\xi, \tag{3.3}$$

where $\Omega_1 = 1+\xi - z_2\xi,\ \Omega_2 = 1+\xi - z\xi,\ \Lambda = \frac{(1+\xi)^2}{\Omega_1\Omega_2}$,

$$(iv)\ H_{A,p,\upsilon}(w_1,w_2,w_3;w_4,w_5;z_1,z_2,z) = \frac{2\Gamma(w_5)}{\Gamma(w_3)\Gamma(w_5-w_3)}\times$$
$$\times\sqrt{\frac{2p}{\pi}}\int_0^{\frac{\pi}{2}}(\sin^2\xi)^{w_3-1}(\cos^2\xi)^{w_5-w_3-1}(1-z_2\sin^2\xi)^{-w_1}(1-z\sin^2\xi)^{-w_2}$$
$$\times K_{\upsilon+\frac{1}{2}}\left(\frac{p}{\sin^2\xi\cos^2\xi}\right){}_2F_1\left(\begin{array}{c} w_1,w_2; \\ w_4; \end{array} Z\right)d\xi, \tag{3.4}$$

where

$$Z = \frac{z_1}{(1-z_2\sin^2\xi)(1-z\sin^2\xi)},$$

$$(v)\ H_{A,p,\upsilon}(w_1,w_2,w_3;w_4,w_5;z_1,z_2,z) = \frac{\sigma_0\ \Gamma(w_5)}{\Gamma(w_3)\Gamma(w_5-w_3)}$$
$$\times\sqrt{\frac{2p}{\pi}}\int_\alpha^\beta \sigma\ \{\sigma_1\}^{-w_1}\{\sigma_2\}^{-w_2}K_{\upsilon+\frac{1}{2}}(\Theta)\,{}_2F_1\left(\begin{array}{c} w_1,w_2; \\ w_4; \end{array} \Delta\ z_1\right)d\xi, \tag{3.5}$$

where

$$\sigma = \frac{(\xi-\alpha)^{w_3-\frac{3}{2}}(\beta-\xi)^{w_5-w_3-\frac{3}{2}}}{(\xi-\gamma)^{w_5-w_1-w_2}-1},$$
$$\sigma_0 = \frac{(\beta-\gamma)^{w_3-\frac{1}{2}}(\alpha-\gamma)^{w_5-w_3-\frac{1}{2}}}{(\beta-\alpha)^{w_5-w_1-w_2-2}},$$
$$\sigma_1 = [(\beta-\alpha)(\xi-\gamma)-z_2(\beta-\gamma)(\xi-\alpha)],$$
$$\sigma_2 = [(\beta-\alpha)(\xi-\gamma)-z(\beta-\gamma)(\xi-\alpha)],$$
$$\Theta = \frac{p(\beta-\alpha)^2(\xi-\gamma)^2}{(\alpha-\gamma)(\beta-\xi)(\beta-\gamma)(\xi-\alpha)},$$
$$\Delta = \frac{(\beta-\alpha)^2(\xi-\gamma)^2}{\sigma_1\sigma_2},$$

and

$$(vi)\ H_{A,p,\upsilon}(w_1,w_2,w_3;w_4,w_5;z_1,z_2,z) = \frac{\Gamma(w_5)(1+\lambda)^{w_3-\frac{1}{2}}}{\Gamma(w_3)\Gamma(w_5-w_3)}$$
$$\times\sqrt{\frac{2p}{\pi}}\int_0^1 \nabla\ \{\nabla_1\}^{-w_1}\{\nabla_2\}^{-w_2}K_{\upsilon+\frac{1}{2}}(\omega)\ {}_2F_1\left(\begin{array}{c} w_1,w_2; \\ w_4; \end{array} \Xi\ z_1\right)d\xi, \tag{3.6}$$

where

$$\nabla = \frac{(\xi)^{w_3-\frac{3}{2}}(1-\xi)^{w_5-w_3-\frac{3}{2}}}{(1+\lambda\xi)^{w_5-w_1-w_2-1}},\ \nabla_1 = [1+\lambda\xi - z_2(1+\lambda)\xi],$$

$$\nabla_2 = [1+\lambda\xi - z(1+\lambda)\xi],\ \omega = \frac{p(1+\lambda\xi)^2}{\xi(1+\lambda)(1-\xi)},\ (\lambda > -1),$$

and

$$\Xi = \frac{(1+\lambda\xi)^2}{\nabla_1\nabla_2}.$$

Proof. The confirmation of the first integral expression (3.1) appears to follow by using the generalised beta function (1.15) in (2.1), shifting the sequence of integration and summation (since such integral is uniform convergence), and, after modification, using the Gauss hypergeometric function, to get the requried result (3.1). It is possible to directly prove the integral expressions (3.2)-(3.6) by making use of the transformations that are listed below:

$$t = \frac{\xi}{1+\xi},\ \frac{dt}{d\xi} = \frac{1}{(1+\xi)^2},\ (\xi > -1), \tag{3.7}$$

$$t = \sin^2\xi,\ \frac{dt}{d\xi} = 2\sin\xi\cos\xi, \tag{3.8}$$

$$t = \frac{(\beta-\gamma)(\xi-\alpha)}{(\beta-\alpha)(\xi-\gamma)},\ \frac{dt}{d\xi} = \frac{(\beta-\alpha)(\beta-\gamma)(\alpha-\gamma)}{(\beta-\alpha)^2(\xi-\gamma)^2}, \tag{3.9}$$

and

$$t = \frac{(1+\lambda)\xi}{1+\lambda\xi},\ \frac{dt}{d\xi} = \frac{(1+\lambda)}{(1+\lambda\xi)^2}, \tag{3.10}$$

one result at a time in (3.1) to get the right side. □

Theorem 3.2. *Let $\Re(p) > 0$, $\Re(w_j) > 0$ $(j = 1, 3, 4, 5)$, $\Re(w_4) > \Re(w_1) > 0$ and $\Re(w_5) > \Re(w_3) > 0$. Then the following integrals expressions holds true.*

$$(vii)\ H_{A,p,\upsilon}(w_1, w_2, w_3; w_4, w_5; z_1, z_2, z) = \frac{\Gamma(w_4)\Gamma(w_5)\sqrt{2p}}{\Gamma(w_1)\Gamma(w_3)\Gamma(w_4-w_1)\Gamma(w_5-w_3)}$$
$$\times \int_0^1\int_0^1 f_1 \cdot G^{2,1}_{1,2}\left(\frac{2p}{t(1-t)}\middle|\begin{matrix}\frac{1}{2}\\ \upsilon+\frac{1}{2}\ ,-\upsilon-\frac{1}{2}\end{matrix}\right)d\xi dt, \tag{3.11}$$

$$= \frac{\Gamma(w_4)\Gamma(w_5)\cos(\upsilon\pi)\sqrt{2p}}{\Gamma(w_1)\Gamma(w_3)\Gamma(w_4-w_1)\Gamma(w_5-w_3)\pi}$$
$$\times \int_0^1\int_0^1 f_1 \cdot G^{2,1}_{1,2}\left(\frac{2p}{t(1-t)}\middle|\begin{matrix}\frac{1}{2}\\ \upsilon+\frac{1}{2}\ ,-\upsilon-\frac{1}{2}\end{matrix}\right)d\xi dt, \tag{3.12}$$

where

$$f_1 = \xi^{w_1-1}t^{w_3-\frac{3}{2}}(1-\xi)^{w_4-w_1-1}(1-t)^{w_5-w_3-\frac{3}{2}}(1-z_2t)^{-w_1}(1-zt)^{-w_2}\exp\left(\frac{p}{t(1-t)}\right),$$

$$(viii)\ H_{A,p,\upsilon}(w_1, w_2, w_3; w_4, w_5; z_1, z_2, z) = \frac{\Gamma(w_4)\Gamma(w_5)(2p)^{-\mu+\frac{1}{2}}}{\Gamma(w_1)\Gamma(w_3)\Gamma(w_4-w_1)\Gamma(w_5-w_3)\sqrt{\pi}}$$
$$\times \int_0^1\int_0^1 f_2 \cdot G^{2,1}_{1,2}\left(\frac{2p}{t(1-t)}\middle|\begin{matrix}\mu+\frac{1}{2}\\ \mu+\upsilon+\frac{1}{2}\ ,\ \mu-\upsilon-\frac{1}{2}\end{matrix}\right)d\xi dt, \tag{3.13}$$

$$= \frac{\Gamma(w_4)\Gamma(w_5)(p)^{r+\frac{1}{2}}\cos(\upsilon\pi)}{\Gamma(w_1)\Gamma(w_3)\Gamma(w_4-w_1)\Gamma(w_5-w_3)\pi\sqrt{2}}$$
$$\times \int_0^1\int_0^1 f_2 \cdot G^{2,1}_{1,2}\left(\frac{2p}{t(1-t)}\middle|\begin{matrix}\mu+\frac{1}{2}\\ \mu+\upsilon+\frac{1}{2}\ ,\ \mu-\upsilon-\frac{1}{2}\end{matrix}\right)d\xi dt, \tag{3.14}$$

where

$$f_2 = \xi^{w_1-1}t^{w_3+\mu-\frac{3}{2}}(1-\xi)^{w_4-w_1-1}(1-t)^{w_5-w_3+\mu-\frac{3}{2}}(1-z_2t)^{-w_1}(1-zt)^{-w_2}\exp\left(\frac{p}{t(1-t)}\right),$$

and

$$(ix)\ H_{A,p,\upsilon}(w_1,w_2,w_3;w_4,w_5;z_1,z_2,z) = \frac{\Gamma(w_4)\Gamma(w_5)p^{\frac{1}{2}-\mu}2^{\mu-\frac{1}{2}}}{\Gamma(w_1)\Gamma(w_3)\Gamma(w_4-w_1)\Gamma(w_5-w_3)\sqrt{\pi}}$$
$$\times\int_0^1\int_0^1 f_3\cdot G^{2,0}_{0,2}\left(\frac{p^2}{4t^2(1-t)^2}\middle|\ \frac{2\mu+2\upsilon+1}{4},\ \frac{2\mu-2\upsilon-1}{4}\right)d\xi dt, \tag{3.15}$$

$$= \frac{\Gamma(w_4)\Gamma(w_5)4^{\mu-1}(2p)^{\frac{1}{2}-\mu}}{\Gamma(w_1)\Gamma(w_3)\Gamma(w_4-w_1)\Gamma(w_5-w_3)\pi^{\frac{3}{2}}}$$
$$\times\int_0^1\int_0^1 f_3\cdot G^{4,0}_{0,4}\left(\frac{p^4}{(4t)^4(1-t)^4}\middle|\ \frac{2\mu+2\upsilon+1}{8},\ \frac{2\mu+2\upsilon+5}{8}, \frac{2\mu-2\upsilon-1}{8},\ \frac{2\mu-2\upsilon+3}{8}\right)d\xi dt, \tag{3.16}$$

where

$$f_3 = \xi^{w_1-1}t^{w_3+\mu-\frac{3}{2}}(1-\xi)^{w_4-w_1-1}(1-t)^{w_5-w_3+\mu-\frac{3}{2}}(1-z_2t)^{-w_1}(1-zt)^{-w_2},$$

$|\arg(1-z_2)|<\pi$,$|\arg(1-z)|<\pi$, $\upsilon\geq 0$ *and* μ *is a free parameter.*

Proof. The integral expressions (3.11)-(3.16) that were discussed earlier can be obtained by substituting (1.8)-(1.13) into the expression of the generic Srivastava function $H_{A,p,\upsilon}(\cdot)$ found in (3.1). □

4. Mellin transforms for $H_{A,p,\upsilon}(\cdot)$

Definition 4.1. If $f(z_1)$ is a locally integrable function on $(0,\infty)$, then the Mellin transform of $f(z_1)$ is defined by [11, p.193]

$$\Phi(\eta) = \mathcal{M}\left\{f(z_1)\right\}(\eta) = \int_0^\infty z_1^{\eta-1}f(z_1)dz_1 \tag{4.1}$$

the strip of analyticity $E<\Re(\eta)<F$ that characterises an analytic function. For the given function (4.1), we can define its inverse Mellin transform as

$$f(z_1) = \mathcal{M}^{-1}\left\{\Phi(\eta)\right\} = \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} z_1^{-\eta}\Phi(\eta)d\eta,\ (E<d<F). \tag{4.2}$$

Theorem 4.2. *The Mellin transform of the generalised function* $H_{A,p,\upsilon}(\cdot)$*, takes for* $\Re(\eta)>\upsilon>0$*. Then we have*

$$\mathcal{M}\left\{H_{A,p,\upsilon}(w_1,w_2,w_3;w_4,w_5;z_1,z_2,z)\right\}(\eta) = \int_0^\infty P^{\eta-1}H_{A,p,\upsilon}(w_1,w_2,w_3;w_4,w_5;z_1,z_2,z)dp,$$
$$= \frac{2^{\eta-1}}{\sqrt{\pi}}\Gamma\left(\frac{\eta-\upsilon}{2}\right)\Gamma\left(\frac{\eta+\upsilon+1}{2}\right)\ H_A^{(\eta)}(w_1,w_2,w_3;w_4,w_5;z_1,z_2,z), \tag{4.3}$$

where $w_4,w_5\in\mathbb{C}\backslash\mathbb{Z}_0^-$ *and* $H_A^{(\eta)}$ *is defined in (1.4).*

Proof. When the series (2.1) is substituted into the integral on the left side of (4.3), the order of integration is changed (due to the uniform convergence of the integral).

$$\mathcal{M}\left\{H_{A,p,\upsilon}(w_1,w_2,w_3;w_4,w_5;z_1,z_2,z)\right\}(\eta)$$
$$=\sum_{i,j,k=0}^{\infty}\left[\frac{(w_1)_{i+j}(w_2)_{i+k}}{(w_4)_i\ B(w_3,w_5-w_3)}\frac{z_1^i}{i!}\frac{z_2^j}{j!}\frac{z^k}{k!}\left\{\int_0^\infty p^{\eta-1}B_{p,\upsilon}(w_3+j+k,w_5-w_3)\right\}dp\right]. \tag{4.4}$$

Utilizing the generalised Beta function (1.15) in the aforementioned equation (4.4) we get

$$\mathcal{M}\left\{H_{A,p,\upsilon}(w_1,w_2,w_3;w_4,w_5;z_1,z_2,z)\right\}(\eta)$$
$$=\sqrt{2/\pi}\sum_{i,j,k=0}^{\infty}\left[\frac{(w_1)_{i+j}(w_2)_{i+k}}{(w_4)_i\ B(w_3,w_5-w_3)}\frac{z_1^i}{i!}\frac{z_2^j}{j!}\frac{z^k}{k!}\times\right.$$
$$\left.\times\int_0^1 t^{w_3+j+k-\frac{3}{2}}(1-t)^{w_5-w_3-\frac{3}{2}}\left\{\int_0^\infty p^{\eta-\frac{1}{2}}K_{\upsilon+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right)dp\right\}dt\right]. \tag{4.5}$$

Implementation of the result [13, eq.(10.43.19)]

$$\int_0^\infty w^{\eta-\frac{1}{2}}K_{\alpha+\frac{1}{2}}(w)dw=2^{\eta-\frac{3}{2}}\Gamma\left(\frac{\eta-\alpha}{2}\right)\Gamma\left(\frac{\eta+\alpha+1}{2}\right), |\Re(\alpha)|<\Re(\eta), \tag{4.6}$$

executed by the replacement $w=p/t(1-t)$ in the preceding equation (4.5) produces

$$\mathcal{M}\left\{H_{A,p,\upsilon}(w_1,w_2,w_3;w_4,w_5;z_1,z_2,z)\right\}(\eta)=\frac{2^{\eta-1}}{\sqrt{\pi}}\Gamma\left(\frac{\eta-\upsilon}{2}\right)\Gamma\left(\frac{\eta+\upsilon+1}{2}\right)\times$$
$$\times\sum_{i,j,k=0}^{\infty}\left[\frac{(w_1)_{i+j}(w_2)_{i+k}}{(w_4)_i\ B(w_3,w_5-w_3)}\frac{z_1^i}{i!}\frac{z_2^j}{j!}\frac{z^k}{k!}\left\{\int_0^1 t^{w_3+j+k+\eta-1}(1-t)^{w_5-w_3+\eta-1}dt\right\}\right]. \tag{4.7}$$

Using a simple Beta function (1.6) in the above equation (4.7), then we get after simplification the right hand side stated in (4.3). □

Corollary 1. The inverse Mellin transform of $H_{A,p,\upsilon}(\cdot)$ function is given by

$$H_{A,p,\upsilon}(w_1,w_2,w_3;w_4,w_5;z_1,z_2,z)=\mathcal{M}^{-1}\left\{\Phi(\eta)\right\}$$
$$=\frac{\sqrt{\pi}}{i4\pi^2}\int_{d-i\infty}^{d+i\infty}\left(\frac{2}{p}\right)^{\eta}\Gamma\left(\frac{\eta-\upsilon}{2}\right)\Gamma\left(\frac{\eta+\upsilon+1}{2}\right)\ H_A^{(\eta)}(w_1,w_2,w_3;w_4,w_5;z_1,z_2,z)d\eta, \tag{4.8}$$

where $d>\upsilon$.

5. A derivative identity for $H_{A,p,\upsilon}(\cdot)$

Theorem 5.1. *The derivative identity for the function $H_{A,p,\upsilon}(\cdot)$ is as follows:*

$$\frac{\partial^{I+J+K}}{\partial z_1^I\partial z_2^J\partial z^K}\left\{H_{A,p,\upsilon}\left(w_1,w_2,w_3;w_4,w_5;z_1,z_2,z\right)\right\}=\frac{(w_1)_{I+K}(w_2)_{I+J}(w_3)_{J+K}}{(w_4)_I(w_5)_{J+K}}$$
$$\times H_{A,p,\upsilon}(w_1+I+K,w_2+I+J,w_3+J+K;w_4+I,w_5+J+K;z_1,z_2,z), \tag{5.1}$$

where $I,J,K\in\mathbb{N}_0$.

Proof. The succeeding result is obtained by partially differentiating the series for

$$\mathcal{H} \equiv H_{A,p,v}(w_1, w_2, w_3; w_4, w_5; z_1, z_2, z),$$

in (2.1) with respect to z_1.

$$\frac{\partial \mathcal{H}}{\partial z_1} = \sum_{i=1}^{\infty} \sum_{j,k=0}^{\infty} \left[\frac{(w_1)_{i+j}(w_2)_{i+k}}{(w_4)_i} \frac{B_{p,v}(w_3+j+k, w_5-w_3)}{B(w_3, w_5-w_3)} \frac{z_1^{i-1}}{(i-1)!} \frac{z_2^j}{j!} \frac{z^k}{k!} \right]. \tag{5.2}$$

Employing an algebraic property $(\lambda)_{\ell+k} = (\lambda)_\ell (\lambda+\ell)_k$, we have upon setting $i \to i+1$

$$\frac{\partial \mathcal{H}}{\partial z_1} = \frac{w_1\ w_2}{w_4} \sum_{i,j,k=0}^{\infty} \left[\frac{(w_1+1)_{i+j}(w_2+1)_{i+k}}{(w_4+1)_i} \frac{B_{p,v}(w_3+j+k, w_5-w_3)}{B(w_3, w_5-w_3)} \frac{z_1^i}{i!} \frac{z_2^j}{j!} \frac{z^k}{k!} \right], \tag{5.3}$$

$$= \frac{w_1\ w_2}{w_4} H_{A,p,v}\left(w_1+1,\ w_2+1,\ w_3;\ w_4+1, w_5\ ; z_1, z_2, z\right) \tag{5.4}$$

Regular operation of (5.4) then yields for $I = 1, 2, 3, ...$

$$\frac{\partial^I \mathcal{H}}{\partial z_1^I} = \frac{(w_1)_I\ (w_2)_I}{(w_4)_I} H_{A,p,v}\left(w_1+I,\ w_2+I,\ w_3;\ w_4+I, w_5\ ; z_1, z_2, z\right). \tag{5.5}$$

A similar thinking shows that (by differentiate partially with respect to z_2)

$$\frac{\partial^{I+1} \mathcal{H}}{\partial z_1^I \partial z_2} = \frac{(w_1)_I\ (w_2)_I}{(w_4)_I} \times \sum_{i,k=0}^{\infty} \sum_{j=1}^{\infty} \left[\frac{(w_1+I)_{i+j}(w_2+I)_{i+k}}{(w_4+I)_i} \frac{B_{p,v}(w_3+j+k, w_5-w_3)}{B(w_3, w_5-w_3)} \frac{z_1^i}{i!} \frac{z_2^{j-1}}{(j-1)!} \frac{z^k}{k!} \right]. \tag{5.6}$$

Upon setting $j \to j+1$, we obtain

$$\frac{\partial^{I+1} \mathcal{H}}{\partial z_1^I \partial z_2} = \frac{(w_1)_{I+1}\ (w_2)_I\ (w_3)}{(w_4)_I\ (w_5)} \times \sum_{i,j,k \geq 0} \left[\frac{(w_1+I+1)_{i+j}(w_2+I)_{i+k}}{(w_4+I)_i} \frac{B_{p,v}(w_3+1+j+k, w_5-w_3)}{B(w_3+1, w_5-w_3)} \frac{z_1^i}{i!} \frac{z_2^j}{j!} \frac{z^k}{k!} \right],$$

$$= \frac{(w_1)_{I+1}\ (w_2)_I\ (w_3)}{(w_4)_I\ (w_5)} H_{A,p,v}\left(w_1+I+1,\ w_2+I,\ w_3+1;\ w_4+I, w_5+1\ ; z_1, z_2, z\right). \tag{5.7}$$

Reuse of (5.7) with J times then yields

$$\frac{\partial^{I+J} \mathcal{H}}{\partial z_1^I \partial z_2^J} = \frac{(w_1)_{I+J}\ (w_2)_I\ (w_3)_J}{(w_4)_I\ (w_5)_J} \times H_{A,p,v}\left(w_1+I+J,\ w_2+I,\ w_3+J;\ w_4+I, w_5+J\ ;\ z_1, z_2, z\right). \tag{5.8}$$

Similarly, differentiate partially to the above eq. (5.8) with respect to z for the series $H_{A,p,v}(\cdot)$ as we have done before. Repeated differentiation K times with respect to z then readily produces the requried result in (5.1). □

6. An upper bound for $H_{A,p,\upsilon}(\cdot)$

Theorem 6.1. *Let the parameters $w_j \in \mathbb{C}(1 \leq j \leq 3)$ and the complex variables $z_1, z_2, z \in \mathbb{C}$, then the function $H_{A,p,\upsilon}(\cdot)$ is valid.*

$$|H_{A,p,\upsilon}(w_1, w_2, w_3; w_4, w_5; z_1, z_2, z)| < \frac{2^{\upsilon}|p|^{\upsilon+1}}{\sqrt{\pi}\{\Re(p)\}^{2\upsilon+1}}\Gamma\left(\upsilon + \frac{1}{2}\right) H_A^{(\upsilon)}(w_1, w_2, w_3; w_4, w_5; |z_1|, |z_2|, |z|), \quad (6.1)$$

where $\Re(p) > 0,\ \upsilon > 0$; parameters $w_4, w_5 \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $H_A^{(\upsilon)}(\cdot)$ is defined in (1.4).

Proof. The modified Bessel function of the second kind has the following equation [13, Entry (10.32.8)], which is connected to the integral expression of the extension $H_{A,p,\upsilon}(\cdot)$ in (3.1).

$$k_{\upsilon+\frac{1}{2}}(z) = \frac{\sqrt{\pi}\left(\frac{1}{2}z\right)^{\upsilon+\frac{1}{2}}}{\Gamma(\upsilon+1)} \int_1^{\infty} e^{-zt}(t^2-1)^{\upsilon}dt,\ (\upsilon > -1,\ \Re(z) > 0). \quad (6.2)$$

We have $\upsilon > 0$ and $\Re(z) > 0$ in our actual problem. Moreover, we allow $x = \Re(z)$ so that

$$\left|k_{\upsilon+\frac{1}{2}}(z)\right| \leq \frac{\sqrt{\pi}\left(\frac{1}{2}|z|\right)^{\upsilon+\frac{1}{2}}}{\Gamma(\upsilon+1)} \left|\int_1^{\infty} e^{-zt}(t^2-1)^{\upsilon}dt\right| < \frac{\sqrt{\pi}\left(\frac{1}{2}|z|\right)^{\upsilon+\frac{1}{2}}}{\Gamma(\upsilon+1)} \int_0^1 t^{2\upsilon}e^{-xt}dt = \frac{\sqrt{\pi}\left(\frac{1}{2}|z|\right)^{\upsilon+\frac{1}{2}}}{\Gamma(\upsilon+1)} \frac{\Gamma(2\upsilon+1,x)}{x^{2\upsilon+1}}. \quad (6.3)$$

The maximum value of the incomplete gamma function is $\Gamma(a,z)$ [13, (8.2.2)]. However, while this result is computationally sharp when z is real, the incomplete gamma function provides the integral for $H_{A,p,\upsilon}(\cdot)$ really hard-to-bound. Make use of the minimal inequality $\Gamma(2\upsilon+1,x) < \Gamma(2\upsilon+1)$ to simplify (6.3).

$$\left|k_{\upsilon+\frac{1}{2}}(z)\right| < \frac{\sqrt{\pi}\left(\frac{1}{2}|z|\right)^{\upsilon+\frac{1}{2}}}{\Gamma(\upsilon+1)} \frac{\Gamma(2\upsilon+1)}{x^{2\upsilon+1}} = \frac{1}{2}\left(\frac{2|z|}{x^2}\right)^{\upsilon+\frac{1}{2}} \Gamma\left(\upsilon+\frac{1}{2}\right). \quad (6.4)$$

when such gamma function duplication formula is used. The bound (6.4) is less sharp than the bound (6.3). Even so, it can be more easily managed in the expression $H_{A,p,\upsilon}(\cdot)$. Put $z = p/t(1-t)$ in the above formula (6.4), where $t \in (0,1)$ and $\Re(p) > 0$, we achieve.

$$\left|k_{\upsilon+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right)\right| < \frac{1}{2}\left(\frac{2|p|\ t(1-t)}{(\Re(p))^2}\right)^{\upsilon+\frac{1}{2}} \Gamma\left(\upsilon+\frac{1}{2}\right). \quad (6.5)$$

For clarity, we will assume that the parameters $w_j > 0$ $(1 \leq j \leq 5)$ and easy to extend to complex parameters. Then, from (3.1), we get

$$|H_{A,p,\upsilon}(w_1, w_2, w_3; w_4, w_5; z_1, z_2, z)| < \frac{2^{\upsilon}|p|^{\upsilon+1}\Gamma\left(\upsilon+\frac{1}{2}\right)}{\sqrt{\pi}\{\Re(p)\}^{2\upsilon+1}} \times \times \sum_{i,j,k\geq 0} \frac{(w_1)_{i+j}\ (w_2)_{i+k}}{(w_4)_i} \frac{B(w_3+\upsilon+j+k, w_5-w_3+\upsilon)}{B(w_3, w_5-w_3)} \frac{|z_1|^i}{i!} \frac{|z_2|^j}{j!} \frac{|z|^k}{k!}. \quad (6.6)$$

Identify the above series by $H_A^{(\upsilon)}(\cdot)$ function, which is defined in (1.4), yielding the result stated in the above upper bounded function (6.1) to $H_{A,p,\upsilon}(\cdot)$. □

TABLE 2. The numerical approximation values of a generalised Srivastava's triple hypergeometric function (2.1) to $H_{A,p,v}(\cdot)$ and the bound (6.1) for distinct p and v values when indices $i = j = k = 0$; parameters $w_1 = 1/2$, $w_2 = 1/3$, $w_3 = 3/2$, $w_4 = 4$, $w_5 = 7/2$ and variables $z_1 = 1$, $z_2 = 2$, $z = 3$.

p	v	$H_{A,p,v}(\cdot)$	Bound	p	v	$H_{A,p,v}(\cdot)$	Bound
0.05	0.10	0.0472012	1.21502	0.40	0.45	0.0702450	1.20064
0.10	0.15	0.0578520	1.22452	0.45	0.50	0.0678072	1.18942
0.15	0.20	0.0646084	1.22944	0.50	0.55	0.0644989	1.13722
0.20	0.25	0.0688510	1.22944	0.55	0.60	0.0603955	1.16457
0.25	0.30	0.0712244	1.22572	0.60	0.65	0.0555585	1.15132
0.30	0.35	0.0720996	1.21923	0.65	0.70	0.0500386	1.13770
0.35	0.40	0.0717172	1.21070	0.70	0.75	0.0438788	1.12382

In table-2, we compute a numerical approximation values of this generalised hypergeometric function $H_{A,p,v}(\cdot)$ with bounds by Wolfram Mathematica software/computer algebraic software/objected oriented programme.

7. Recurrence relations for $H_{A,p,v}(\cdot)$

We prepare two recurrence relations for the extended Srivastava's triple hypergeometric function $H_{A,p,v}(\cdot)$ in this section. The first process gives a recurrences with respect to the numerator parameters w_1 and w_2, while the second provides recurrence relations for the extended Srivastava's function with respect to the denominator parameter w_4.

Theorem 7.1. *The recurrence relations for the $H_{A,p,v}(\cdot)$ function with the numerator parameters w_1 and w_2 are as follows:*

$$\begin{aligned} H_{A,p,v}(w_1+1, w_2, w_3; w_4, w_5; z_1, z_2, z) &= H_{A,p,v}(w_1, w_2, w_3; w_4, w_5; z_1, z_2, z) + \\ &+ \frac{z_1 w_2}{w_4} H_{A,p,v}(w_1+1, w_2+1, w_3; w_4+1, w_5; z_1, z_2, z) + \\ &+ \frac{z_2 w_3}{w_5} H_{A,p,v}(w_1+1, w_2, w_3+1; w_4, w_5; z_1, z_2, z), \end{aligned} \tag{7.1}$$

and

$$\begin{aligned} H_{A,p,v}(w_1, w_2+1, w_3; w_4, w_5; z_1, z_2, z) &= H_{A,p,v}(w_1, w_2, w_3; w_4, w_5; z_1, z_2, z) + \\ &+ \frac{z_1 w_1}{w_4} H_{A,p,v}(w_1+1, w_2+1, w_3; w_4+1, w_5; z_1, z_2, z) + \\ &+ \frac{z w_3}{w_5} H_{A,p,v}(w_1, w_2+1, w_3+1; w_4, w_5; z_1, z_2, z). \end{aligned} \tag{7.2}$$

Proof. In light of the fact that (2.1) and the subsequent result $(w_1+1)_{i+j} = (w_1)_{i+j}(1+i/w_1+j/w_1)$, we derive

$$H_{A,p,v}(w_1+1, w_2, w_3; w_4, w_5; z_1, z_2, z) = \sum_{i,j,k=0}^{\infty}\left[\frac{(w_1+1)_{i+j}(w_2)_{i+k}}{(w_4)_i}\frac{B_{p,v}(w_3+j+k, w_5-w_3)}{B(w_3, w_5-w_3)}\frac{z_1^i}{i!}\frac{z_2^j}{j!}\frac{z^k}{k!}\right], \quad (7.3)$$

$$= H_{A,p,v}(w_1, w_2, w_3; w_4, w_5; z_1, z_2, z) + \frac{z_1}{w_1}\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left[\frac{(w_1)_{i+j}(w_2)_{i+k}}{(w_4)_i}\frac{B_{p,v}(w_3+j+k, w_5-w_3)}{B(w_3, w_5-w_3)}\frac{z_1^{i-1}}{(i-1)!}\frac{z_2^j}{j!}\frac{z^k}{k!}\right] + \frac{z_2}{w_1}\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\left[\frac{(w_1)_{i+j}(w_2)_{i+k}}{(w_4)_i}\frac{B_{p,v}(w_3+j+k, w_5-w_3)}{B(w_3, w_5-w_3)}\frac{z_1^i}{i!}\frac{z_2^{j-1}}{(j-1)!}\frac{z^k}{k!}\right]. \quad (7.4)$$

Take a look at the first sum in (7.4), which will be denoted by the letter S from now on. To find the answer, replace $i \to i+1$ and use the identity $(w)_{j+1} = w(w+1)_j$.

$$S = \frac{z_1 w_2}{w_4}\sum_{i,j,k\geq 0}\left[\frac{(w_1+1)_{i+j}(w_2+1)_{i+k}}{(w_4+1)_i}\frac{B_{p,v}(w_3+j+k, w_5-w_3)}{B(w_3, w_5-w_3)}\frac{z_1^i}{i!}\frac{z_2^j}{j!}\frac{z^k}{k!}\right]. \quad (7.5)$$

$$= \frac{z_1 w_2}{w_4} H_{A,p,v}(w_1+1, w_2+1, w_3; w_4+1, w_5; z_1, z_2, z). \quad (7.6)$$

Continuing along the same lines for the second series in (7.4) with $j \to j+1$, we arrive at the conclusion that this sum can be represented as

$$\frac{z_2 w_3}{w_5} H_{A,p,v}(w_1+1, w_2, w_3+1; w_4, w_5; z_1, z_2, z). \quad (7.7)$$

The conclusion reached in (7.1) can be attained by first combining the findings from (7.6) and (7.7) with those from (7.4).
In a similar manner, the proof of (7.2) can be obtained in the same manner by exchanging w_2 and making use of the one more fact, which is described by

$$B(w_3, w_5-w_3) = \frac{w_5}{w_3} B(w_3+1, w_5-w_3), \quad (7.8)$$

Then we get after simplification the result stated in (7.2). □

Corollary 2. Following is a recursion that is valid based on the given reference (7.1).

$$H_{A,p,v}(w_1+J, w_2, w_3; w_4, w_5; z_1, z_2, z) = H_{A,p,v}(w_1, w_2, w_3; w_4, w_5; z_1, z_2, z) + \frac{z_1 b_2}{w_4}\sum_{\ell=1}^{J} H_{A,p,v}(w_1+\ell, w_2+1, w_3; w_4+1, w_5; z_1, z_2, z) + \frac{w_3 z_2}{w_5}\sum_{\ell=1}^{J} H_{A,p,v}(w_1+\ell, w_2, w_3+1; w_4, w_5; z_1, z_2, z). \quad (7.9)$$

for positive integer J.

Corollary 3. Following recursion holds true according to the given equation (7.2)

$$H_{A,p,v}(w_1, w_2 + J, w_3; w_4, w_5; z_1, z_2, z) = H_{A,p,v}(w_1, w_2, w_3; w_4, w_5; z_1, z_2, z) + \\ + \frac{z_1\ w_1}{w_4} \sum_{\ell=1}^{J} H_{A,p,v}(w_1 + 1, w_2 + \ell, w_3; w_4 + 1, w_5; z_1, z_2, z) + \\ + \frac{w_3\ z}{w_5} \sum_{\ell=1}^{J} H_{A,p,v}(w_1, w_2 + \ell, w_3 + 1; w_4, w_5; z_1, z_2, z). \quad (7.10)$$

for positive integer J.

Theorem 7.2. *For the function $H_{A,p,v}(\cdot)$, the next recurrence holds correct with regard to the denominator parameter w_4*

$$H_{A,p,v}(w_1, w_2, w_3; w_4, w_5; z_1, z_2, z) = H_{A,p,v}(w_1, w_2, w_3; w_4 + 1, w_5; z_1, z_2, z) \\ + \frac{z_1 w_1 w_2}{w_4(w_4 + 1)} H_{A,p,v}(w_1 + 1, w_2 + 1, w_3; w_4 + 2, w_5; z_1, z_2, z). \quad (7.11)$$

Proof. Look into the case in which w_4 is decreased by 1, to be specific

$$\mathfrak{F} = H_{A,p,v}(w_1, w_2, w_3; w_4 - 1, w_5; z_1, z_2, z)$$

, and use the identity $(w_4 - 1)_i = (w_4)_i / \{1 + \frac{i}{w_4 - 1}\}$. Then

$$\mathfrak{F} = \sum_{i,j,k \geq 0} \left[\frac{(w_1)_{i+j}(w_2)_{i+k}}{(w_4 - 1)_i} \frac{B_{p,v}(w_3 + j + k, w_5 - w_3)}{B(w_3, w_5 - w_3)} \frac{z_1^i}{i!} \frac{z_2^j}{j!} \frac{z^k}{k!} \right],$$

$$= \sum_{i,j,k \geq 0} \left[\frac{(w_1)_{i+j}(w_2)_{i+k}}{(w_4)_i} \frac{B_{p,v}(w_3 + j + k, w_5 - w_3)}{B(w_3, w_5 - w_3)} \left(1 + \frac{i}{w_4 - 1}\right) \frac{z_1^i}{i!} \frac{z_2^j}{j!} \frac{z^k}{k!} \right],$$

$$= H_{A,p,v}(w_1, w_2, w_3; w_4, w_5; z_1, z_2, z) \\ + \frac{z_1}{w_4 - 1} \sum_{i=1}^{\infty} \sum_{j,k=0}^{\infty} \left[\frac{(w_1)_{i+j}(w_2)_{i+k}}{(w_4)_i} \frac{B_{p,v}(w_3 + j + k, w_5 - w_3)}{B(w_3, w_5 - w_3)} \frac{z_1^{i-1}}{(i-1)!} \frac{z_2^j}{j!} \frac{z^k}{k!} \right]. \quad (7.12)$$

When we factor in $i \to i + 1$ into the estimation given above, we get the following

$$\mathfrak{F} = H_{A,p,v}(w_1, w_2, w_3; w_4, w_5; z_1, z_2, z) \\ + \frac{z_1\ w_1 w_2}{w_4(w_4 - 1)} \sum_{i,j,k \geq 0} \left[\frac{(w_1 + 1)_{i+j}(w_2 + 1)_{i+k}}{(w_4 + 1)_i} \frac{B_{p,v}(w_3 + j + k, w_5 - w_3)}{B(w_3, w_5 - w_3)} \frac{z_1^i}{i!} \frac{z_2^j}{j!} \frac{z^k}{k!} \right] \quad (7.13)$$

$$= H_{A,p,v}(w_1, w_2, w_3; w_4, w_5; z_1, z_2, z) \\ + \frac{z_1 w_1 w_2}{w_4(w_4 - 1)} H_{A,p,v}(w_1 + 1, w_2 + 1, w_3; w_4 + 1, w_5; z_1, z_2, z). \quad (7.14)$$

Afterwards, the result that is obtained by changing w_4 by $w_4 + 1$ can be found in (7.11). □

Corollary 4. Following recursion holds true according to the given equation (7.11)

$$H_{A,p,\upsilon}(w_1, w_2, w_3; w_4 + J, w_5; z_1, z_2, z) = \sum_{\ell=1}^{J} H_{A,p,\upsilon}(w_1, w_2, w_3; w_4 + \ell, w_5; z_1, z_2, z)$$

$$+ \frac{z_1 w_1 w_2}{w_4(w_4+1)} \sum_{\ell=2}^{J} H_{A,p,\upsilon}(w_1 + 1, w_2 + 1, w_3; w_4 + \ell, w_5; z_1, z_2, z), \quad (7.15)$$

for positive integer J.

8. Conclusion

A generalised Srivastava's triple hypergeometric function $H_{A,p,\upsilon}(\cdot)$ has been presented here by us. In addition to this, we provided certain characteristics of this series, including the Mellin transforms, a derivative identity, a bounded inequality, and recurrence relations. In addition, we have justified specific integral expressions of the function $H_{A,p,\upsilon}(\cdot)$ by involving Meijer's G-function and Gauss hypergeometric functions in the process. In addition, we compute numerical approximation table of this generalised triple hypergeometric function $H_{A,p,\upsilon}(\cdot)$ with bounds by Wolfram mathematica software / computer algebraic software. The numerical approximation table for the upper bound can also be used in computer algorithm programming, which is one possible application. Further, we can generalised Srivastava's triple hypergeometric function to investigate certain problems, namely, to solve some differential equation, evaluate integrals, and analyse certain physical phenomena.

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