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Research article

Short-time free metaplectic transform: Its relation to short-time Fourier transform in $L^2(\mathbb{R}^n)$ and uncertainty principles

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Abstract: The free metaplectic transformation (FMT) has gained much popularity in recent times because of its various applications in signal processing, paraxial optical systems, digital algorithms, optical encryption and so on. However, the FMT is inadequate for localized analysis of non-transient signals, as such, it is imperative to introduce a unique localized transform coined as the short-time free metaplectic transform (ST-FMT). In this paper, we investigate the ST-FMT. First we propose the definition of the ST-FMT and provide the time-frequency analysis of the proposed transform in the FMT domain. Second we establish the relationship between the ST-FMT and short-time Fourier transform (STFT) in $L^2(\mathbb{R}^n)$ and investigate the basic properties of the proposed transform including the reconstruction formula, Moyal's formula. The emergence of the ST-FMT definition and its properties broadens the development of time-frequency representation of higher-dimensional signals theory to a certain extent. We extend some different uncertainty principles (UPs) from quantum mechanics including Lieb's inequality, Pitt's inequality, Hausdorff-Young inequality, Heisenberg's UP, Hardy's UP, Beurling's UP, Logarithmic UP and Nazarov's UP. Finally, we give a numerical example and a possible applications of the proposed ST-FMT.

Keywords: short-time free metaplectic transform; Moyal's formula; uncertainty principle; Nazarov's UP; Hardy's UP; Logarithmic's UP
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1. Introduction

The free metaplectic transformation (FMT) also known as the n-dimensional linear canonical transformation (LCT) first studied in [1], is widely used in many fields such as filter design, pattern recognition, optics and analyzing the propagation of electromagnetic waves [2–5]. The theory of

FMT with a general $2n \times 2n$ real, symplectic matrix $\mathbf{M} = (A, B : C, D)$ with $n(2n+1)$ degrees of freedom [6, 7]. The FMT embodies several signal processing tools ranging from the classical Fourier, Fresnel transform, and even the fundamental operations of quadratic phase factor multiplication [8–10]. The metaplectic operator or the FMT of any function $f \in L^2(\mathbb{R}^n)$ with the real free symplectic matrix $\mathbf{M} = (A, B : C, D)$ is given by [9, 32]

$$\mathcal{L}_{\mathbf{M}}[f](w) = \int_{\mathbb{R}^n} f(x)\mathcal{K}_{\mathbf{M}}(x, w)dx, \tag{1.1}$$

where $\mathcal{K}_{\mathbf{M}}(x, w)$ denotes the kernel and is given by

$$\mathcal{K}_{\mathbf{M}}(x, w) = \frac{1}{(2\pi)^{n/2}\sqrt{|det(B)|}} e^{\frac{i}{2}(w^T DB^{-1}w - 2w^T B^{-T}x + x^T B^{-1}Ax)}, \quad x, w \in \mathbb{R}^n, |det(B)| \neq 0. \tag{1.2}$$

The arbitrary real parameters involved in (1.1) are of great importance for the efficient analysis of the inescapable signals, i.e., the chirp-like signals. Due to the extra degrees of freedom, the FMT has been successfully employed in diverse problems arising in various branches of science and engineering, such as harmonic analysis, optical systems, reproducing kernel Hilbert spaces, quantum mechanics, image processing, sampling and so on [11, 12]. However, FMT has a drawback. Due to its global kernel it is not suitable for processing signals with varying frequency content [13, 14]. The short-time free metaplectic transform (ST-FMT) with a local window function can efficiently localize the frequency spectrum of non-transient signals in the free metaplectic transform domain, hence, overcomes this drawback. Taking this opportunity, our goal is to introduce the notion of the short-time free metaplectic transform, which is a generalized version of FMT and endowed with higher degrees of freedom, resulting in an efficient localized analysis of chirp signals.

Let us now move to the uncertainty iprinciple. Uncertainty principle was introduced by the German physicist Heisenberg [15] in 1927 and is known as the heart of any signal processing tool. With the passage of time, researchers further extended the uncertainty principle to different types of new uncertainty principles associated with the Fourier transform, for instance, Heisenberg's uncertainty principle, Logarithmic uncertainty principle, Hardy's uncertainty principle and Beurling's uncertainty principle [16–21]. Later these uncertainty principles were extended to LCT and its generalized domains [22–31]. In [32] authors proposed uncertainty principles associated with the FMT and in [33] authors establish uncertainty principles for the non-separable linear canonical wavelet transform. Keeping in view the fact that the theory of ST-FMT and associated uncertainty principles is yet to be investigated exclusively; therefore, it is both theoretically interesting and practically useful to study the properties of ST-FMT and formulate some new uncertainty inequalities pertaining to it.

The highlights of the article are pointed out below:

(1) To introduce a novel integral transform coined as the short-time free metaplectic transform.
(2) To obtain the relationship of ST-FMT with FMT, FT and STFT in $L^2(\mathbb{R}^n)$,respectively.
(3) To study the fundamental properties of the proposed transform, including the Moyal's formula, boundedness and inversion formula.
(4) To establish the Pitt's inequality, Lieb inequality and Hausdorff-Young inequality associated with the ST-FMT.
(5) To formulate the Heisenberg's, Logarithmic and Nazarov's uncertainty principles.
(6) To formulate the Hardy's and Beurling's uncertainty principles for the ST-FMT.

(7) To present possible applications and an illustrative example.

The paper is organized as follows: In Section 2, we discuss some preliminary results and definitions required in subsequent sections. In Section 3, we formally introduced the definition of short-time free metaplectic transform (ST-FMT). Then we investigated several basic properties of the ST-FMT that are important for signal representation in signal processing. In Section 4, we extend some different uncertainty principles (UP) from quantum mechanics, including Lieb's inequality, Pitt's inequality, Heisenberg's UP, Hausdorff-Young inequality, Hardy's UP, Beurling's UP, Logarithmic UP and Nazarov's UP, which have already been well studied in the FMT domain. Section 5 is devoted to a numerical example and potential applications of the proposed ST-FMT.

2. Preliminary

This section give some useful definitions and lemmas about the multi-dimensional Fourier transform and free metaplectic transform.

2.1. *Multi-dimensional Fourier transform (FT)*

For any $f \in L^2(\mathbb{R}^n)$, the n-dimensional Fourier transform (FT) of $f(x)$ is given by [32]

$$\mathcal{F}[f](w) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-iw^T x} dx \tag{2.1}$$

and its inversion is given by

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \mathcal{F}[f](w) e^{iw^T x} dw, \tag{2.2}$$

where $x = (x_1, x_2, ..., x_n)^T \in \mathbb{R}^n$, $w = (w_1, w_2, ..., w_n)^T \in \mathbb{R}^n$.

Based on the definition of short-time Fourier transform (STFT) [34], we can define n-dimensional STFT as:

Definition 2.1. Let ϕ be a window function in $L^2(\mathbb{R}^n)$, then for any $f \in L^2(\mathbb{R}^n)$ the n-dimensional short-time Fourier transform (STFT) of $f(x)$ with respect to the window function ϕ is given by [33]

$$\mathcal{V}_\phi[f](w,u) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) \overline{\phi(x-u)} e^{-iw^T x} dx. \tag{2.3}$$

2.2. *The free metaplectic transform*

For typographical convenience, we shall denote a $2n \times 2n$ matrix $\mathbf{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ as $\mathbf{M} = (A, B : C, D)$, where A, B, C and D are real $n \times n$ sub-matrices. Moreover, we recall that the matrix $\mathbf{M}$ is said to be a free symplectic matrix if $\mathbf{M}^T \mathbf{\Omega} \mathbf{M} = \mathbf{\Omega}$, and $|det(B)| \neq 0$, where $\mathbf{\Omega} = (0, I_n : -I_n, 0)$ and I_n represents the $n \times n$ identity matrix. The sub-matrices of $\mathbf{M}$ satisfy the following constraints:

$$AB^T = BA^T, \quad CD^T = DC^T, \quad AD^T - BC^T = I_n.$$

The transpose of $\mathbf{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is given by $\mathbf{M}^T = \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix}$.

Also, inverse of the free symplectic matrix is given by $\mathbf{M}^{-1} = \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix}$. It is clear that $\mathbf{MM^{-1}} = \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix}$.

Definition 2.2. [FMT] The free metaplectic transform of a function $f \in L^2(\mathbb{R}^n)$ with respect $2n \times 2n$ real, free symplectic matrix $\mathbf{M} = (A, B : C, D)$ (with $|det(B)| \neq 0$) is defined by

$$\mathcal{L}_{\mathbf{M}}[f](w) = \int_{\mathbb{R}^n} f(x)\mathcal{K}_{\mathbf{M}}(x, w)dx, \tag{2.4}$$

where $\mathcal{K}_{\mathbf{M}}(x, w)$ denotes the kernel and is given by

$$\mathcal{K}_{\mathbf{M}}(x, w) = \frac{1}{(2\pi)^{n/2}\sqrt{|det(B)|}} e^{\frac{i}{2}(w^T DB^{-1}w - 2w^T B^{-T}x + x^T B^{-1}Ax)}, \tag{2.5}$$

where $x = (x_1, x_2, ..., x_n)^T \in \mathbb{R}^n$, $w = (w_1, w_2, ..., w_n)^T \in \mathbb{R}^n$.

Definition 2.3. Suppose $f \in L^2(\mathbb{R}^n)$, then the inversion of the free metaplectic transform of f is given by

$$f(x) = \mathcal{L}_{\mathbf{M}^{-1}}\left\{\mathcal{L}_{\mathbf{M}}[f](w)\right\}(x) = \int_{\mathbb{R}^n} \mathcal{L}_{\mathbf{M}}[f](w)\mathcal{K}_{\mathbf{M}^{-1}}(w, x)dw, \tag{2.6}$$

where $\mathbf{M}^{-1} = (D^T, -B^T : -C^T, A^T)$.

For real, symplectic matrix $\mathbf{M} = (A, B : C, D)$, the free metaplectic transform kernel (2.5) satisfies the following important properties:

(i) $\mathcal{K}_{\mathbf{M}^{-1}}(w, x) = \overline{\mathcal{K}_{\mathbf{M}}(x, w)}$,

(ii) $\int_{\mathbb{R}^n} \mathcal{K}_{\mathbf{M}}(x, w)\mathcal{K}_{\mathbf{M}^{-1}}(z, x)dx = \delta(z - w)$,

(iii) $\int_{\mathbb{R}^n} \mathcal{K}_{\mathbf{M}}(x, w)\mathcal{K}_{\mathbf{N}}(x, z)dx = \mathcal{K}_{\mathbf{MN}}(w, z)$.

Lemma 2.1. *Let $f, g \in L^2(\mathbb{R}^n)$, then the free metaplectic transform satisfies the following parseval's formula:*

$$\langle f, g\rangle_2 = \langle \mathcal{L}_{\mathbf{M}}[f], \mathcal{L}_{\mathbf{M}}[g]\rangle_2. \tag{2.7}$$

For $f = g$, one has

$$\|f\|^2_{L^2(\mathbf{R}^n)} = \|\mathcal{L}_{\mathbf{M}}[f]\|^2_{L^2(\mathbf{R}^n)}. \tag{2.8}$$

The free metaplectic transform (defined in (2.4)) of a function $f \in L^2(\mathbb{R}^n)$ can be computed via associated n-dimensional FT, namely

$$\mathcal{L}_{\mathbf{M}}[f](w) = \frac{e^{\frac{i(w^T DB^{-1}w)}{2}}}{\sqrt{det(B)}} \mathcal{F}\left\{e^{\frac{i(x^T B^{-1}Ax)}{2}} f(x)\right\}(B^{-1}w), \tag{2.9}$$

where $\mathcal{F}\{f\}(w)$ represents the n-dimensional FT defined in (2.1) as

$$\mathcal{F}\{f\}(w) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x)e^{-iw^T x}dx. \tag{2.10}$$

Equation (2.9) can be rewritten as

$$e^{\frac{-i(w^T DB^{-1}w)}{2}}\mathcal{L}_{\mathbf{M}}[f](w) = \mathcal{F}\{H\}(B^{-1}w), \tag{2.11}$$

where $H(x)$ is given by

$$H(x) = \frac{1}{\sqrt{det(B)}}e^{\frac{i(x^T B^{-1}Ax)}{2}}f(x). \tag{2.12}$$

Lemma 2.2. *Let $f \in L^2(\mathbb{R}^n)$ and $H(x) = \frac{1}{\sqrt{det(B)}}e^{\frac{i(x^T B^{-1}Ax)}{2}}f(x)$, then we have following relationship between n-dimensional FT and FMT:*

$$|\mathcal{F}[H](w)| = |\mathcal{L}_{\mathbf{M}}[f](Bw)|. \tag{2.13}$$

Proof. The proof follows by changing $w = Bw$ and taking the modulus in (2.11).

□

Remark 2.1. If we change $w = Bw$ and take the modulus in (2.9), we have

$$|\mathcal{F}[H_0](w)| = \sqrt{det(B)}|\mathcal{L}_{\mathbf{M}}[f](Bw)|, \tag{2.14}$$

where $H_0(x) = e^{\frac{i(x^T B^{-1}Ax)}{2}}f(x)$.

3. The short-time free metaplectic transform

In this section, we introduce the novel short-time free metaplectic transform (ST-FMT) and discuss several basic properties of the ST-FMT. These properties play important roles in signal representation of multi-dimensional signals.

Definition 3.1. Let $\mathbf{M} = (A, B : C, D)$ be a real, free symplectic matrix. Let ϕ be a window function in $L^2(\mathbb{R}^n)$, the short-time free metaplectic transform (ST-FMT) of the function $f \in L^2(\mathbb{R}^n)$ with respect to ϕ is defined by

$$\mathcal{V}_\phi^{\mathbf{M}}[f](w, u) = \int_{\mathbb{R}^n} f(x)\overline{\phi(x-u)}\mathcal{K}_{\mathbf{M}}(x, w)dx, \tag{3.1}$$

where $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, $u = (u_1, u_2, ..., u_n) \in \mathbb{R}^n$, $w = (w_1, w_2, ..., w_n) \in \mathbb{R}^n$ and $\mathcal{K}_{\mathbf{M}}(x, w)$ is given by (2.5).

In terms of classic convolution, formula (3.1) can be written as

$$\mathcal{V}_\phi^{\mathbf{M}}[f](w, u) = (f(u)\mathcal{K}_{\mathbf{M}}(u, w)) * \overline{\phi(-u)}.$$

Also, by applying the properties of the free metaplectic transform, (3.1) can be rewritten in the form of an inner product as

$$\mathcal{V}_\phi^{\mathbf{M}}[f](w, u) = \langle f, \Psi_{x,w,u}^{\mathbf{M}}\rangle, \tag{3.2}$$

where $\Psi_{x,w,u}^{\mathbf{M}} = \phi(x-u)\mathcal{K}_{\mathbf{M}}(u, w)$.

The prolificacy of the short-time free metaplectic transform given in Definition 3.1 can be ascertained from the following important deductions:

(i) When $\mathbf{M} = (I_n \cos\alpha, I_n \sin\alpha : -I_n \sin\alpha, I_n \cos\alpha)$, the ST-FMT (3.1) yields the n-dimensional non-separable short-time fractional Fourier transform:

$$\mathcal{F}_{\phi}^{\alpha}[f](w,u) = \frac{1}{(2\pi)^{n/2}|\sin\alpha|^{n/2}} \int_{\mathbb{R}^n} f(x)\overline{\phi(x-u)} e^{\frac{i}{2}(w^T w + x^T x)\cot\alpha - i w^T x \csc\alpha} dx.$$

(ii) For $\mathbf{M} = (\mathbf{0}, I_n \ \ -I_n, \mathbf{0})$, the ST-FMT (3.1) boils down to an n-dimensional short-time Fourier transform:

$$\mathcal{F}_{\phi}[f](w,u) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x)\overline{\phi(x-u)} e^{-iw^T x} dx.$$

(iii) If the sub-matrices $A = diag(a_{11}, ..., a_{nn})$, $B = diag(b_{11}, ..., b_{nn})$, $C = diag(c_{11}, ..., c_{nn})$ and $D = diag(d_{11}, ..., d_{nn})$ of the real, symplectic matrix $\mathbf{M}$ are taken then ST-FMT (3.1) yields the short-time separable linear canonical transform:

$$\mathcal{L}_{\phi}^{\mathbf{M}}[f](w,u) = \frac{1}{(2\pi)^{n/2}|\Pi_{j=1}^{n} b_{jj}|^{1/2}} \int_{\mathbb{R}^n} f(x)\overline{\phi(x-u)} e^{i\sum_{j=1}^{n}\left(\frac{d_{jj}w_j^2 - 2w_j x_j + a_{jj}x_j^2}{2b_{jj}}\right)} dx.$$

(iv) For $\mathbf{M} = (I_n, B : \mathbf{0}, I_n)$, the ST-FMT (3.1) boils down to an n-dimensional short-time Fresnel transform:

$$\mathcal{F}_{\phi}^{\mathbf{M}}[f](w,u) = \frac{1}{(2\pi)^{n/2}\sqrt{|det(B)|}} \int_{\mathbb{R}^n} f(x)\overline{\phi(x-u)} e^{i\frac{(w^T B^{-1} w - 2w^T B^{-T} x + x^T B^{-1} x)}{2}} dx.$$

Now we shall establish a relationship between FMT and ST-FMT, which will be helpful in establishing various uncertainty principles in Section 4.

For fixed u, we have

$$\mathcal{V}_{\phi}^{\mathbf{M}}[f](w,u) = \mathcal{L}_{\mathbf{M}}\left\{f(x)\overline{\phi(x-u)}\right\}(w). \tag{3.3}$$

Applying the inverse free metaplectic transform (2.6), we have

$$\begin{aligned} f(x)\overline{\phi(x-u)} &= \mathcal{L}_{\mathbf{M}^{-1}}\left\{\mathcal{V}_{\phi}^{\mathbf{M}}[f](w,u)\right\} & (3.4)\\ &= \int_{\mathbb{R}^n} \mathcal{V}_{\phi}^{\mathbf{M}}[f](w,u)\mathcal{K}_{\mathbf{M}^{-1}}(w,x)dw \\ &= \int_{\mathbb{R}^n} \mathcal{V}_{\phi}^{\mathbf{M}}[f](w,u)\overline{\mathcal{K}_{\mathbf{M}}(x,w)}dw. & (3.5)\end{aligned}$$

Now, we discuss several basic properties of the ST-FMT given by (3.1). These properties play important roles in multi-dimensional signal processing.

The following properties follow directly from the definition of the ST-FMT, viz: linearity; anti-linearity; translation; modulation and scaling are omitted. We shall focus on boundedness, Moyal's formula and inversion.

Lemma 3.1. *(Relation with n-dimensional STFT)*

$$\begin{aligned} &\mathcal{V}_{\phi}^{\mathbf{M}}[f](w,u) \\ &= \frac{1}{(2\pi)^{n/2}\sqrt{|det(B)|}} \int_{\mathbb{R}^n} f(x)\overline{\phi(x-u)} e^{\frac{i(w^T DB^{-1} w + x^T B^{-1} Ax - 2w^T B^{-T} x)}{2}} dx\end{aligned}$$

$$= \frac{e^{i\frac{w^T DB^{-1}w}{2}}}{(2\pi)^{n/2}\sqrt{|det(B)|}} \int_{\mathbb{R}^n} e^{\frac{i(x^T B^{-1}Ax)}{2}} f(x)\overline{\phi(x-u)} e^{-i(B^{-1}w)^T x} dx$$
$$= e^{i\frac{w^T DB^{-1}w}{2}} \mathcal{V}_\phi[H](B^{-1}w, u),$$

where $\mathcal{V}_\phi[f]$ represents an n-dimensional STFT given in Definition 2.1.

And

$$H(x) = \frac{1}{\sqrt{|det(A)|}} e^{\frac{i(x^T B^{-1}Ax)}{2}} f(x). \tag{3.6}$$

Next we prove the lemma that is very important in establishing various uncertainty principles in Section 4.

Lemma 3.2. *Let $f \in L^2(\mathbb{R}^n)$ and $F(x) = e^{\frac{i(x^T B^{-1}Ax)}{2}} f(x)\overline{\phi(x-u)}$, then we have following relationship between n-dimensional FT and ST-FMT:*

$$|\mathcal{F}[F](w)| = \sqrt{|det(B)|}|\mathcal{V}^{\mathbf{M}}_\phi[f](Bw, u)|. \tag{3.7}$$

Proof. From (3.1), we have

$$\mathcal{V}^{\mathbf{M}}_\phi[f](w, u)$$
$$= \frac{1}{(2\pi)^{n/2}\sqrt{|det(B)|}} \int_{\mathbb{R}^n} f(x)\overline{\phi(x-u)} e^{\frac{i(w^T DB^{-1}w + x^T B^{-1}Ax - 2w^T B^{-T} x)}{2}} dx$$
$$= \frac{e^{i\frac{w^T DB^{-1}w}{2}}}{(2\pi)^{n/2}\sqrt{|det(B)|}} \int_{\mathbb{R}^n} e^{\frac{i(x^T B^{-1}Ax)}{2}} f(x)\overline{\phi(x-u)} e^{-i(B^{-1}w)^T x} dx$$
$$= \frac{1}{\sqrt{|det(B)|}} e^{i\frac{w^T DB^{-1}w}{2}} \mathcal{F}[F](B^{-1}w).$$

On changing $w = Bw$ and taking the modulus in (3.8), we get

$$\sqrt{|det(B)|}|\mathcal{V}^{\mathbf{M}}_\phi[f](Bw, u)| = |\mathcal{F}[F](w)|,$$

which completes the proof. □

Remark 3.1. If we take $F_0(x) = \frac{e^{\frac{i(x^T B^{-1}Ax)}{2}}}{\sqrt{det(B)}} f(x)\overline{\phi(x-u)}$, then the above lemma yields

$$|\mathcal{F}[F_0](w)| = |\mathcal{L}_{\mathbf{M}}[f](Bw)|. \tag{3.8}$$

Theorem 3.2. *(Boundedness) Let $f, \phi \in L^2(\mathbb{R}^n)$, where ϕ is a non-zero window function, then we have*

$$\left|\mathcal{V}^{\mathbf{M}}_\phi[f](w, u)\right| \leq \frac{1}{(2\pi)^{n/2}\sqrt{|det(B)|}} \|f\|_{L^2(\mathbb{R}^n)} \|\phi\|_{L^2(\mathbb{R}^n)}.$$

Proof. By using the Cauchy-Schwarz inequality in Definition 3.1, we have

$$\begin{aligned}
&\left|\mathcal{V}_{\phi}^{\mathbf{M}}[f](w,u)\right|^2\\
&= \left|\int_{\mathbb{R}^n} f(x)\overline{\phi(x-u)}\mathcal{K}_{\mathbf{M}}(x,w)dx\right|^2\\
&\leq \left(\int_{\mathbb{R}^n} \left|f(x)\overline{\phi(x-u)}\mathcal{K}_{\mathbf{M}}(x,w)\right| dx\right)^2\\
&= \left(\frac{1}{(2\pi)^{n/2}\sqrt{|det(B)|}}\int_{\mathbb{R}^n} \left|f(x)\overline{\phi(x-u)}e^{\frac{i(w^T DB^{-1}w+x^T B^{-1}Ax-2w^T B^{-T}x)}{2}}\right| dx\right)^2\\
&= \left(\frac{1}{(2\pi)^{n/2}\sqrt{|det(B)|}}\int_{\mathbb{R}^n} \left|f(x)\overline{\phi(x-u)}\right| dx\right)^2\\
&= \frac{1}{(2\pi)^{n}|det(B)|}\left(\int_{\mathbb{R}^n} \left|f(x)\overline{\phi(x-u)}\right| dx\right)^2\\
&= \frac{1}{(2\pi)^{n}|det(B)|}\left(\int_{\mathbb{R}^n} |f(x)|^2 dx\right)\left(\int_{\mathbb{R}^n} \left|\overline{\phi(x-u)}\right|^2 dx\right)\\
&= \frac{1}{(2\pi)^{n}|det(B)|}\|f\|^2_{L^2(\mathbb{R}^n)}\|\phi\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

On further simplification, we obtain

$$\left|\mathcal{V}_{\phi}^{\mathbf{M}}[f](w,u)\right| \leq \frac{1}{(2\pi)^{n/2}\sqrt{|det(B)|}}\|f\|_{L^2(\mathbb{R}^n)}\|\phi\|_{L^2(\mathbb{R}^n)},$$

which completes the proof. □

In the following theorem, we show that the proposed ST-FMT is reversible in the sense that the input signal f can be recovered easily from the transformed domain.

Theorem 3.3. *(Inversion) For any fixed window function $\phi \in L^2(\mathbb{R}^n)$. Then, any $f \in L^2(\mathbb{R}^n)$ can be reconstructed by the formula*

$$f(x) = \frac{1}{\|\phi(x)\|^2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathcal{V}_{\phi}^{\mathbf{M}}[f](w,u)\phi(x-u)\mathcal{K}_{\mathbf{M}^{-1}}(x,w)dwdu. \tag{3.9}$$

Proof. From (3.4), we have

$$\begin{aligned}
f(x)\overline{\phi(x-u)} &= \mathcal{L}_{\mathbf{M}^{-1}}\left\{\mathcal{V}_{\phi}^{\mathbf{M}}[f](w,u)\right\}\\
&= f(x)\overline{\phi(x-u)}\\
&= \int_{\mathbb{R}^n} \mathcal{V}_{\phi}^{\mathbf{M}}[f](w,u)\mathcal{K}_{\mathbf{M}^{-1}}(x,w)dw.
\end{aligned}$$

Multiplying both sides of the last equation by $\phi(x-u)$ and integrating with respect to du, yields

$$\int_{\mathbb{R}^n} f(x)\overline{\phi(x-u)}\phi(x-u)du = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathcal{V}_{\phi}^{\mathbf{M}}[f](w,u)\mathcal{K}_{\mathbf{M}^{-1}}(x,w)\phi(x-u)dwdu$$

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)|\phi(x-u)|^2 du &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathcal{V}_\phi^{\mathbf{M}}[f](w,u)\phi(x-u)\mathcal{K}_{\mathbf{M}^{-1}}(x,w)dwdu\\ f(x)\|\phi(x)\|^2 &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathcal{V}_\phi^{\mathbf{M}}[f](w,u)\phi(x-u)\mathcal{K}_{\mathbf{M}^{-1}}(x,w)dwdu,\end{aligned}$$

which implies

$$f(x) = \frac{1}{\|\phi(x)\|^2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathcal{V}_\phi^{\mathbf{M}}[f](w,u)\phi(x-u)\mathcal{K}_{\mathbf{M}^{-1}}(x,w)dwdu.$$

□

Theorem 3.4. *(Moyal's formula) Let $\mathcal{V}_{\phi_1}^{\mathbf{M}}[f](w,u$ and $\mathcal{V}_{\phi_2}^{\mathbf{M}}[g](w,u)$ be the ST-FMT transforms of f and g, respectively. Then, we have*

$$\left\langle \mathcal{V}_{\phi_1}^{\mathbf{M}}[f](w,u), \mathcal{V}_{\phi_2}^{\mathbf{M}}[g](w,u)\right\rangle = \langle f,g\rangle_{L^2(\mathbb{R}^n)}\langle \phi_1,\phi_2\rangle_{L^2(\mathbb{R}^n)}. \tag{3.10}$$

Proof.

$$\begin{aligned}&\left\langle \mathcal{V}_{\phi_1}^{\mathbf{M}}[f](w,u), \mathcal{V}_{\phi_2}^{\mathbf{M}}[g](w,u)\right\rangle\\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathcal{V}_{\phi_1}^{\mathbf{M}}[f](w,u)\mathcal{V}_{\phi_2}^{\mathbf{M}}[g](w,u)dwdu\\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathcal{V}_{\phi_1}^{\mathbf{M}}[f](w,u)\left[\overline{\int_{\mathbb{R}^n} g(x)\overline{\phi_2(x-u)}\mathcal{K}_{\mathbf{M}}(x,w)dx}\right]dwdu\\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathcal{V}_{\phi_1}^{\mathbf{M}}[f](w,u)\overline{\mathcal{K}_{\mathbf{M}}(x,w)}.\overline{g(x)}\phi_2(x-u)dxdudw\\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n} \mathcal{V}_{\phi_1}^{\mathbf{M}}[f](w,u)\overline{\mathcal{K}_{\mathbf{M}}(x,w)}dw\right]\overline{g(x)}\phi_2(x-u)dxdu.\end{aligned}$$

Now applying (3.4) in the above equation, we obtain

$$\begin{aligned}\left\langle \mathcal{V}_{\phi_1}^{\mathbf{M}}[f](w,u), \mathcal{V}_{\phi_2}^{\mathbf{M}}[g](w,u)\right\rangle &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)\overline{\phi_1(x-u)}.\overline{g(x)}\phi_2(x-u)dxdu\\ &= \int_{\mathbb{R}^n} f(x)\overline{g(x)}dx\int_{\mathbb{R}^n}\overline{\phi_1(x-u)}\phi_2(x-u)du\\ &= \langle f,g\rangle_{L^2(\mathbb{R}^n)}\langle \phi_1,\phi_2\rangle_{L^2(\mathbb{R}^n)},\end{aligned}$$

which completes the proof. □

From the above theorem, we obtain the following consequences:
(i) If $\phi_1 = \phi_2$, then

$$\left\langle \mathcal{V}_{\phi}^{\mathbf{M}}[f](w,u), \mathcal{V}_{\phi}^{\mathbf{M}}[g](w,u)\right\rangle = \|\phi\|_{L^2(\mathbb{R}^n)}\langle f,g\rangle_{L^2(\mathbb{R}^n)}. \tag{3.11}$$

(ii) If $f = g$ and $\phi_1 = \phi_2$, then

$$\left\langle \mathcal{V}_\phi^{\mathbf{M}}[f](w,u), \mathcal{V}_\phi^{\mathbf{M}}[f](w,u) \right\rangle = \|\phi\|_{L^2(\mathbb{R}^n)} \|f\|_{L^2(\mathbb{R}^n)}. \tag{3.12}$$

(iii) If $f = g$ and $\phi_1 = \phi_2 = 1$, then

$$\left\langle \mathcal{V}_\phi^{\mathbf{M}}[f](w,u), \mathcal{V}_\phi^{\mathbf{M}}[f](w,u) \right\rangle = \|f\|_{L^2(\mathbb{R}^n)}. \tag{3.13}$$

Equation (3.13) states that the proposed ST-FMT (3.1) becomes an isometry from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$. In other words, the total energy of a signal computed in the short-time non-seperable linear canonical domain is equal to the total energy computed in the spatial domain.

Remark 3.5. Suppose that $\|f\|^2_{L^2(\mathbb{R}^n)} = 1$ and $\|g\|^2_{L^2(\mathbb{R}^n)} = 1$. Then (3.12) gives

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathcal{V}_\phi^{\mathbf{M}}[f](w,u) dw du = 1. \tag{3.14}$$

Equation (3.14) is known as the radar uncertainty principle in the ST-FMT domain. It is easily seen that the function $\mathcal{V}_\phi^{\mathbf{M}}[f](w,u)$ cannot be concentrated arbitrarily close to the origin.

4. N-dimensional uncertainty principles for the short-time free metaplectic transform

The uncertainty principle lies at the heart of harmonic analysis, which asserts that "the position and the velocity of a particle cannot be determined precisely at the same time." Authors in [32] studied various uncertainty principles associated with non-seperable LCT. Since ST-FMT is the generalized version of FMT and keeping in view the fact that the theory of UP's for the short-time free metaplectic transform is yet to be explored exclusively, it is natural and interesting to study different forms of UP's in the ST-FMT domain.

4.1. Pitt's inequality

The Pitt's inequality in the Fourier domain expresses a fundamental relationship between a sufficiently smooth function and the corresponding Fourier transform [26]. The Pitt's inequality introduced in [32] can be extended to the free metaplectic transformation domain. We here extend it to the ST-FMT domain.

Lemma 4.1. *(Pitt's inequality for the FMT [35]) Let $f \in \mathbb{S}(\mathbb{R}^n)$ the Schwartz class in $L^2(\mathbb{R}^n)$, then we have the inequality*

$$\int_{\mathbb{R}^n} |w|^{-\alpha} |\mathcal{L}_{\mathbf{M}}[f](w)|^2 dw \le C_\alpha |det(B)|^{-\alpha} \int_{\mathbb{R}^n} |x|^\alpha |f(x)|^2 dx,$$

where $C_\alpha = \pi^\alpha \left[\Gamma\left(\frac{n-\alpha}{4}\right) / \Gamma\left(\frac{n+\alpha}{4}\right)\right]^2$ and $0 \le \alpha < n$.

Based on Pitt's inequality for the FMT, we obtain Pitt's inequality for the ST-FMT.

Theorem 4.1. *(Pitt's inequality for the ST-FMT) Under the assumptions of Lemma 4.1, we have*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |w|^{-\alpha} |\mathcal{V}_\phi^{\mathbf{M}}[f](w,u)|^2 dw du \tag{4.1}$$

$$= C_\alpha |det(B)|^{-\alpha} \|\phi\|^2_{L^2(\mathbb{R}^n)} \int_{\mathbb{R}^n} |x|^\alpha |f(x)|^2 dx. \tag{4.2}$$

Proof. Let $f_u^{\phi}(x) = f(x)\overline{\phi(x-u)}$, then by virtue of (3.3), we have

$$\int_{\mathbb{R}^n} |w|^{-\alpha} |\mathcal{V}_{\phi}^{\mathbf{M}}[f](w,u)|^2 dw = \int_{\mathbb{R}^n} |w|^{-\alpha} |\mathcal{L}_{\mathbf{M}}[f_u^{\phi}(x)]|^2 dw.$$

Now applying Lemma 4.1 to the above equation, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^n} |w|^{-\alpha} |\mathcal{V}_{\phi}^{\mathbf{M}}[f](w,u)|^2 dw \\ &\leq C_\alpha |det(B)|^{-\alpha} \int_{\mathbb{R}^n} |x|^{\alpha} |f_u^{\phi}(x)|^2 dx \\ &\leq C_\alpha |det(B)|^{-\alpha} \int_{\mathbb{R}^n} |x|^{\alpha} |f(x)\overline{\phi(x-u)}|^2 dx. \end{aligned} \tag{4.3}$$

On integrating both sides of (4.3) with respect to du, we have

$$\begin{aligned} &\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |w|^{-\alpha} |\mathcal{V}_{\phi}^{\mathbf{M}}[f](w,u)|^2 dw du \\ &\leq C_\alpha |det(B)|^{-\alpha} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |x|^{\alpha} |f(x)\overline{\phi(x-u)}|^2 dx du \\ &= C_\alpha |det(B)|^{-\alpha} \|\phi\|^2_{L^2(\mathbb{R}^n)} \int_{\mathbb{R}^n} |x|^{\alpha} |f(x)|^2 dx, \end{aligned}$$

which completes the proof. □

4.2. Lieb's UP

Here we shall establish Lieb's uncertainty principle for the ST-FMT by using the relation between ST-FMT with the short-time Fourier transform in $L^2(\mathbb{R}^n)$.

Theorem 4.2. *(Lieb's) Let $\phi, f \in L^2(\mathbb{R}^n)$ and $2 \leq p < \infty$. Then the following inequality holds:*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\mathcal{V}_{\phi}^{\mathbf{M}}[f](w,u)|^p dw dx \leq \frac{2}{p} \frac{|B|}{|det(A)|^{p/2}} (\|f\|_{L^2(\mathbb{R}^n)} \|\phi\|_{L^2(\mathbb{R}^n)})^p.$$

Proof. For every $f \in \mathbb{S}(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n)$, the Leib's inequality for the STFT states that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\mathcal{V}_{\phi}[f](w,u)|^p dw dx \leq \frac{2}{p} \left(\|f\|_{L^2(\mathbb{R}^n)} \|\phi\|_{L^2(\mathbb{R}^n)}\right)^p, \tag{4.4}$$

where $\mathcal{V}_{\phi}[f](w,u)$ denotes the STFT of f given by (2.1) and $\mathbb{S}(\mathbb{R}^n)$ is the Schwartz class in $L^2(\mathbb{R}^n)$. To obtain an analogue of the Leib's inequality for the ST-FMT, we replace f in the above equation by H (defined in 2.12),

$$\begin{aligned} &\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\mathcal{V}_{\phi}[H](w,u)|^p dw dx \\ &\leq \frac{2}{p} \left(\|H\|_{L^2(\mathbb{R}^n)} \|\phi\|_{L^2(\mathbb{R}^n)}\right)^p \end{aligned}$$

$$= \frac{2}{p}\left(\left(\left(\int_{\mathbb{R}^n}\left|\frac{1}{\sqrt{|det(A)|}}e^{\frac{i(x^TB^{-1}Ax)}{2}}f(x)\right|^2 dx\right)^{1/2}\|\phi\|_{L^2(\mathbb{R}^n)}\right)^p. \tag{4.5}$$

Setting $w = B^{-1}w$ in (4.5), we get

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|B^{-1}||\mathcal{V}_\phi[H](B^{-1}w,u)|^p dwdx\\ &\leq \frac{2}{p}\frac{1}{|det(A)|^{p/2}}\left(\left(\left(\int_{\mathbb{R}^n}|f(x)|^2 dx\right)^{1/2}\|\phi\|_{L^2(\mathbb{R}^n)}\right)^p.\end{aligned} \tag{4.6}$$

Now using Lemma 3.1, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|B^{-1}||e^{\frac{-iw^TDB^{-1}w}{2}}\mathcal{V}_\phi^{\mathbf{M}}[f](w,u)|^p dwdx\\ &\leq \frac{2}{p}\frac{1}{|det(A)|^{p/2}}\left(\left(\left(\int_{\mathbb{R}^n}|f(x)|^2 dx\right)^{1/2}\|\phi\|_{L^2(\mathbb{R}^n)}\right)^p.\end{aligned}$$

On further simplifying, we have

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\mathcal{V}_\phi^{\mathbf{M}}[f](w,u)|^p dwdx\\ &\leq \frac{2|B|}{p}\frac{1}{|det(A)|^{p/2}}\left(\left(\left(\int_{\mathbb{R}^n}|f(x)|^2 dx\right)^{1/2}\|\phi\|_{L^2(\mathbb{R}^n)}\right)^p\\ &= \frac{2}{p}\frac{|B|}{|det(A)|^{p/2}}(\|f\|_{L^2(\mathbb{R}^n)}\|\phi\|_{L^2(\mathbb{R}^n)})^p,\end{aligned}$$

which completes the proof.

□

4.3. Heisenberg's UP

In [32], authors introduced Heisenberg's UP to observe the lower bound of uncertainty corresponding to the free metaplectic transformation. We extend it to the proposed Heisenberg's UP for the ST-FMT.

Theorem 4.3. *(Heisenberg's UP) Let $\mathcal{V}_\phi^{\mathbf{M}}[f]$ be the short-time free metaplectic transform of any non-trivial function $f \in L^2(\mathbb{R}^n)$ with respect to a real free symplectic matrix $\mathbf{M} = (A, B : C, D)$, then the following inequality holds:*

$$\begin{aligned}&\left\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|w|^2\left|\mathcal{V}_\phi^{\mathbf{M}}[f](w,u)\right|^2 dw\right\}^{1/2}\left\{\int_{\mathbb{R}^n}x^2|f(x)|^2 dx\right\}^{1/2}\\ &\geq \frac{n\sigma_{min}(B)}{4\pi}\|f\|^2_{L^2(\mathbb{R}^n)}\|\phi\|_{L^2(\mathbb{R}^n)}.\end{aligned} \tag{4.7}$$

Proof. For any $f \in L^2(\mathbb{R}^n)$ the Heisenberg–Pauli–Weyl uncertainty inequality for the FMT domain is given by [32]:

$$\int_{\mathbb{R}^n} x^2|f(x)|^2dx \int_{\mathbb{R}^n} |w|^2|\mathcal{L}_{\mathbf{M}}[f](w)|^2dw \geq \frac{n^2\sigma_{min}^2(B)}{16\pi^2}\left\{\int_{\mathbb{R}^n} |f(x)|^2dx\right\}^2.$$

Using the inversion of FMT (2.6) into the (2.8), above inequality becomes

$$\int_{\mathbb{R}^n} x^2\,|\mathcal{L}_{\mathbf{M}^{-1}}\{\mathcal{L}_{\mathbf{M}}[f](w)\}|^2\,dx \int_{\mathbb{R}^n} |w|^2\,|\mathcal{L}_{\mathbf{M}}[f](w)|^2\,dw$$
$$\geq \frac{n^2\sigma_{min}^2(B)}{16\pi^2}\left\{\int_{\mathbb{R}^n} |\mathcal{L}_{\mathbf{M}}[f](w)|^2\,dw\right\}^2,$$

since $\mathcal{V}_\phi^{\mathbf{M}}[f] \in L^2(\mathbb{R}^n)$, therefore replacing $\mathcal{L}_{\mathbf{M}}[f](w)$ by $\mathcal{V}_\phi^{\mathbf{M}}[f] \in L^2(\mathbb{R}^n)$, we obtain

$$\int_{\mathbb{R}^n} x^2\left|\mathcal{L}_{\mathbf{M}^{-1}}\{\mathcal{V}_\phi^{\mathbf{M}}[f](w,u)\}\right|^2dx \int_{\mathbb{R}^n} |w|^2\left|\mathcal{V}_\phi^{\mathbf{M}}[f](w,u)\right|^2dw$$
$$\geq \frac{n^2\sigma_{min}^2(B)}{16\pi^2}\left\{\int_{\mathbb{R}^n} \left|\mathcal{V}_\phi^{\mathbf{M}}[f](w,u)\right|^2dw\right\}^2.$$

First taking the square of the above equation and then integrating both sides with respect du, we obtain

$$\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n} x^2\left|\mathcal{L}_{\mathbf{M}^{-1}}\{\mathcal{V}_\phi^{\mathbf{M}}[f](w,u)\}\right|^2dx\right\}^{1/2}\left\{\int_{\mathbb{R}^n} |w|^2\left|\mathcal{V}_\phi^{\mathbf{M}}[f](w,u)\right|^2dw\right\}^{1/2}du$$
$$\geq \frac{n\sigma_{min}(B)}{4\pi}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left|\mathcal{V}_\phi^{\mathbf{M}}[f](w,u)\right|^2dwdu.$$

As a consequence of the Cauchy–Schwartz's inequality and Fubini theorem, the above inequality yields

$$\left\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} x^2\left|\mathcal{L}_{\mathbf{M}^{-1}}\{\mathcal{V}_\phi^{\mathbf{M}}[f](w,u)\}\right|^2dxdu\right\}^{1/2}\left\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |w|^2\left|\mathcal{V}_\phi^{\mathbf{M}}[f](w,u)\right|^2dw\right\}^{1/2}$$
$$\geq \frac{n\sigma_{min}(B)}{4\pi}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left|\mathcal{V}_\phi^{\mathbf{M}}[f](w,u)\right|^2dwdu.$$

Now using (3.4) and (3.12) the above equation, we have

$$\left\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} x^2\left|f(x)\overline{\phi(x-u)}\right|^2dxdu\right\}^{1/2}\left\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |w|^2\left|\mathcal{V}_\phi^{\mathbf{M}}[f](w,u)\right|^2dw\right\}^{1/2}$$
$$\geq \frac{n\sigma_{min}(B)}{4\pi}||\phi||^2_{L^2(\mathbb{R}^n)}||f||^2_{L^2(\mathbb{R}^n)}.$$

Simplifying, we obtain

$$\left\{||\phi||^2_{L^2(\mathbb{R}^n)}\int_{\mathbb{R}^n} x^2\,|f(x)|^2\,dx\right\}^{1/2}\left\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |w|^2\left|\mathcal{V}_\phi^{\mathbf{M}}[f](w,u)\right|^2dw\right\}^{1/2}$$
$$\geq \frac{n\sigma_{min}(B)}{4\pi}||\phi||^2_{L^2(\mathbb{R}^n)}||f||^2_{L^2(\mathbb{R}^n)}.$$

On dividing both sides by $||\phi||_{L^2(\mathbb{R}^n)}$, we will get the desired result. □

Theorem 4.4. *(Hausdorff-Young) Let* $1 \le p \le 2$ *and* q *be such that* $\frac{1}{p} + \frac{1}{q} = 1$*. Then for* $\phi \in L^q(\mathbb{R}^n)$ *and* $f \in L^p(\mathbb{R}^n)$*, the following inequality holds:*

$$\|\mathcal{V}_\phi^{\mathbf{M}}[f](w,u)\|_{L^q(\mathbb{R}^n)} \le \|\phi\|_{L^q(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)}. \tag{4.8}$$

Proof. The definition of FMT 2.2 and procedure defined in Theorem 5.1 [36] together leads to

$$\|\mathcal{L}_{\mathbf{M}}[f]\|_{L^q(\mathbb{R}^n)} \le \|f\|_{L^p(\mathbb{R}^n)}. \tag{4.9}$$

For $p = 1$, we get

$$\|\mathcal{L}_{\mathbf{M}}[f]\|_{L^\infty(\mathbb{R}^n)} \le \|f\|_{L^1(\mathbb{R}^n)}. \tag{4.10}$$

Now using (4.10) and taking $\|\phi\|_{L^q(\mathbb{R}^n)} = 1$, Eq (3.3) yields

$$\begin{aligned}\|\mathcal{V}_\phi^{\mathbf{M}}[f](w,u)\|_{L^\infty(\mathbb{R}^n)} &= \|\mathcal{L}_{\mathbf{M}}\{f(x)\overline{\phi(x-u)}\}\|_{L^\infty(\mathbb{R}^n)}\\ &\le \|\{f(x)\overline{\phi(x-u)}\}\|_{L^1(\mathbb{R}^n)}\\ &\le \|f\|_{L^1(\mathbb{R}^n)}\|\phi\|_{L^\infty(\mathbb{R}^n)}\\ &= \|f\|_{L^1(\mathbb{R}^n)}.\end{aligned}$$

Now for $p = 2$, we obtain

$$\|\mathcal{V}_\phi^{\mathbf{M}}[f](w,u)\|_{L^2(\mathbb{R}^n)} \le \|f\|_{L^2(\mathbb{R}^n)}.$$

By Riesz-Thorin interpolation theorem, we have

$$\|\mathcal{V}_\phi^{\mathbf{M}}[f](w,u)\|_{L^q(\mathbb{R}^n)} \le \|f\|_{L^p(\mathbb{R}^n)}. \tag{4.11}$$

Now setting $g = \frac{\phi}{\|\phi\|_{L^q(\mathbb{R}^n)}}$, where ϕ is a window function in $L^p(\mathbb{R}^n)$, we have by anti-linearity property of FMT

$$\mathcal{V}_g^{\mathbf{M}}[f](w,u) = \frac{1}{\|\phi\|_{L^q(\mathbb{R}^n}}\mathcal{V}_\phi^{\mathbf{M}}[f](w,u). \tag{4.12}$$

On taking $\phi = g$ in (4.11), we obtain

$$\|\mathcal{V}_g^{\mathbf{M}}[f](w,u)\|_{L^q(\mathbb{R}^n)} \le \|f\|_{L^p(\mathbb{R}^n)}, \tag{4.13}$$

which on simplification becomes

$$\|\mathcal{V}_\phi^{\mathbf{M}}[f](w,u)\|_{L^q(\mathbb{R}^n)} \le \|\phi\|_{L^q(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)}.$$

This completes the proof. □

4.4. Logarithmic UP

In 1995, W. Beckner [26] introduced the Logarithmic uncertainty principle. In this subsection we obtain the concept of Logarithmic uncertainty principle for the short-time free metaplectic transform as follows:

Theorem 4.5. *(Logarithmic UP) Let ϕ be a window function in $L^2(\mathbb{R}^n)$ and let $\mathcal{V}_\phi^{\mathbf{M}}[f](w,u) \in \mathbb{S}(\mathbb{R}^n)$, then the short-time free metaplectic transform satisfies the following logarithmic uncertainty inequality:*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \ln|wB^{-T}||\mathcal{V}_\phi^{\mathbf{M}}[f](w,u)|^2 dwdu + \|\phi\|^2_{L^2(\mathbb{R}^n)}\int_{\mathbb{R}^n} \ln|x||f(x)|^2 dx$$
$$\geq \left(\frac{\Gamma'(n/2)}{\Gamma(n/2)} - \ln\pi\right)\|\phi\|^2_{L^2(\mathbb{R}^n)}\|f\|^2_{L^2(\mathbb{R}^n)}.$$

Proof. For any $f \in \mathbb{S}(\mathbb{R}^n)$, the Logarithmic UP for the free metaplectic transform domain is given by [32]:

$$\int_{\mathbb{R}^n} \ln|x||f(x)|^2 dx + \int_{\mathbb{R}^n} \ln|wB^{-T}||\mathcal{L}_{\mathbf{M}}[f](w)|^2 dw$$
$$\geq \left(\frac{\Gamma'(n/2)}{\Gamma(n/2)} - \ln\pi\right)\int_{\mathbb{R}^n} |f(x)|^2 dx.$$

Now invoking the inversion formula of the free metaplectic transform and the Parseval's formula, we obtain

$$\int_{\mathbb{R}^n} \ln|x||\mathcal{L}_{\mathbf{M}^{-1}}\{\mathcal{L}_{\mathbf{M}}[f](w)\}|^2 dx + \int_{\mathbb{R}^n} \ln|wB^{-T}||\mathcal{L}_{\mathbf{M}}[f](w)|^2 dw$$
$$\geq \left(\frac{\Gamma'(n/2)}{\Gamma(n/2)} - \ln\pi\right)\int_{\mathbb{R}^n} |\mathcal{L}_{\mathbf{M}}[f](w)|^2 dw.$$

Since both $\mathcal{L}_{\mathbf{M}}[f](w)$ and $\mathcal{V}_\phi^{\mathbf{M}}[f](w,u)$ are in $\mathbb{S}(\mathbb{R}^n)$, we can replace $\mathcal{L}_{\mathbf{M}}[f](w)$ by $\mathcal{V}_\phi^{\mathbf{M}}[f](w,u)$ on both sides to get

$$\int_{\mathbb{R}^n} \ln|x||\mathcal{L}_{\mathbf{M}^{-1}}\{\mathcal{V}_\phi^{\mathbf{M}}[f](w,u)\}|^2 dx + \int_{\mathbb{R}^n} \ln|wB^{-T}||\mathcal{V}_\phi^{\mathbf{M}}[f](w,u)|^2 dw$$
$$\geq \left(\frac{\Gamma'(n/2)}{\Gamma(n/2)} - \ln\pi\right)\int_{\mathbb{R}^n} |\mathcal{V}_\phi^{\mathbf{M}}[f](w,u)|^2 dw.$$

Integrating above inequality with respect du on both sides and then by virtue of Fubini's theorem, we have

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \ln|x||\mathcal{L}_{\mathbf{M}^{-1}}\{\mathcal{V}_\phi^{\mathbf{M}}[f](w,u)\}|^2 dxdu + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \ln|wB^{-T}||\mathcal{V}_\phi^{\mathbf{M}}[f](w,u)|^2 dwdu$$
$$\geq \left(\frac{\Gamma'(n/2)}{\Gamma(n/2)} - \ln\pi\right)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\mathcal{V}_\phi^{\mathbf{M}}[f](w,u)|^2 dwdu.$$

Now using (3.4) and (3.12) in thef above inequality, we have

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \ln|wB^{-T}||\mathcal{V}_\phi^{\mathbf{M}}[f](w,u)|^2 dwdu + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \ln|x||f(x)\phi(x-u)|^2 dxdu$$
$$\geq \left(\frac{\Gamma'(n/2)}{\Gamma(n/2)} - \ln\pi\right)\|\phi\|^2_{L^2(\mathbb{R}^n)}\|f\|^2_{L^2(\mathbb{R}^n)}.$$

Further simplifying, we obtain the desired result as

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \ln|wB^{-T}||\mathcal{V}_\phi^{\mathbf{M}}[f](w,u)|^2 dwdu + \|\phi\|^2_{L^2(\mathbb{R}^n)}\int_{\mathbb{R}^n}\ln|x||f(x)|^2dx$$
$$\geq \left(\frac{\Gamma'(n/2)}{\Gamma(n/2)} - \ln\pi\right)\|\phi\|^2_{L^2(\mathbb{R}^n)}\|f\|^2_{L^2(\mathbb{R}^n)}.$$

□

4.5. Hardy's UP

G. H. Hardy first introduced the Hardy's uncertainty principle in 1933 [37]. Hardy's uncertainty principle says that it is impossible for a function and its Fourier transform to decrease very rapidly simultaneously. Hardy's UP in the Fourier transform domain [38] was given as follows:

Lemma 4.2. *(Hardy's UP in the Fourier transform [38]) If* $f(x) = O(e^{-|x|^2/\beta^2})$, $\mathcal{F}[f](w) = O((2\pi)^{n/2}e^{-16\pi^2|w|^2/\alpha^2})$ *and* $1/\alpha\beta > 1/4$, *then* $f \equiv 0$. *If* $1/\alpha\beta = 1/4$, *then*

$$f = Ce^{-|x|^2/\beta^2},$$

where C is a constant in $\mathbb{C}$.

Based on Lemma 4.2, we derive the corresponding Hardy's UP for the ST-FMT.

Theorem 4.6. *(Hardy's UP in the ST-FMT) If* $f(x) = O(e^{-|x|^2/\beta^2})$, $\mathcal{F}[f](w) = O((2\pi)^{n/2}e^{-16\pi^2|B^{-1}w|^2/\alpha^2})$ *and* $1/\alpha\beta > 1/4$, *then* $f \equiv 0$. *If* $1/\alpha\beta = 1/4$, *then*

$$f = Ce^{-|x|^2/\beta^2 - \frac{i(x^TB^{-1}Ax)}{2}},$$

where C is a constant in $\mathbb{C}$.

Proof. Consider then function $F(x) = e^{\frac{i(x^TB^{-1}Ax)}{2}}f(x)\overline{\phi(x-u)}$.
On setting $u = x$ then we have

$$|F(x)| = |f(x)||\overline{\phi(0)}| = O(e^{-|x|^2/\beta^2}).$$

Also by Lemma 3.2, we have

$$\begin{aligned} |\mathcal{F}[F](w)| &= \sqrt{det(B)}|\mathcal{V}_\phi^{\mathbf{M}}[f](Bw,u)| \\ &= O((2\pi)^{n/2}e^{-16\pi^2|B^{-1}w|^2/\alpha^2}). \end{aligned}$$

Following from Lemma 4.2:
If $1/\alpha\beta > 1/4$, we have $F \equiv 0$, which implies $f \equiv 0$.
Also, if $1/\alpha\beta = 1/4$, then

$$F(x) = Ce^{-|x|^2/\beta^2}$$

implies

$$f(x) = Ce^{-|x|^2/\beta^2 - \frac{i(x^TB^{-1}Ax)}{2}}.$$

□

This completes the proof.

4.6. Beurling's UP

Beurling's uncertainty principle is a more general version of Hardy's uncertainty principle, which is given by A. Beurling. It implies the weak form of Hardy's UP. Beurling's UP in the Fourier transform domain is as follows:

Lemma 4.3. *(Beurling's UP in the Fourier transform domain [24]) Let $f \in L^2(\mathbb{R}^n)$ and $d \geq 0$ satisfy*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)||\mathcal{F}[f](w)|}{(1+\|x\|+\|w\|)^d} e^{2\pi|\langle x,w\rangle|} dxdw \leq \infty.$$

Then

$$f(x) = P(x)e^{-\pi\langle Ax,x\rangle},$$

where A is a real positive definite symmetric matrix and $P(x)$ is a polynomial of degree $< \frac{d-n}{2}$.

According to Lemma 4.3, we derive Beurling's uncertainty principle for the ST-FMT.

Theorem 4.7. *(Beurling's UP) Let $f, \phi \in L^2(\mathbb{R}^n)$ where ϕ is a nonzero window function and $\mathcal{V}_\phi^{\mathbf{M}}[f] \in L^2(\mathbb{R}^n)$ satisfies*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)||\overline{\phi(x-u)}||\mathcal{V}_\phi^{\mathbf{M}}[f](w,u)|}{\sqrt{|det(B)|}(1+\|x\|+\|B^{-1}w\|)^d} e^{2\pi|\langle x,B^{-1}w\rangle|} dxdw < \infty, \quad \textit{where} \quad d \geq 0.$$

Then

$$f(x) = P(x)/\overline{\phi(x-u)}e^{\frac{-i(x^TB^{-1}Ax)}{2}-\pi\langle Ax,x\rangle},$$

where A is a real positive definite symmetric matrix and $P(x)$ is a polynomial of degree $< \frac{d-n}{2}$.

Proof. Consider the function $F(x) = e^{\frac{i(x^TB^{-1}Ax)}{2}} f(x)\overline{\phi(x-u)}$, then

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|F(x)||\mathcal{F}[F](w)|}{(1+\|x\|+\|w\|)^d} e^{2\pi|\langle x,w\rangle|} dxdw \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)||\overline{\phi(x-u)}|\sqrt{|det(B)|}|\mathcal{V}_\phi^{\mathbf{M}}[f](Bw,u)|}{(1+\|x\|+\|w\|)^d} e^{2\pi|\langle x,w\rangle|} dxdw \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)||\overline{\phi(x-u)}||\mathcal{V}_\phi^{\mathbf{M}}[f](w,u)|}{\sqrt{|det(B)|}(1+\|x\|+\|B^{-1}w\|)^d} e^{2\pi|\langle x,B^{-1}w\rangle|} dxdw < \infty.
\end{aligned}$$

Therefore by Lemma 4.3, we have

$$F(x) = P(x)e^{-\pi\langle Ax,x\rangle},$$

where A is a real positive definite symmetric matrix and $P(x)$ is a polynomial of degree $< \frac{d-n}{2}$. Furthermore $f(x) = P(x)/\overline{\phi(x-u)}e^{\frac{-i(x^TB^{-1}Ax)}{2}-\pi\langle Ax,x\rangle}$. □

This completes the proof.

4.7. Nazarov's UP

Nazarov's UP was first proposed by F. L. Nazarov in 1993 [22]. It measures the localization of a nonzero function by taking into consideration the notion of support of the function instead of the dispersion. In other words it argues what happens if a non-trival function and its Fourier transform are only small outside a compact set. Let us start with Nazarov's UP for the Fourier transform.

Lemma 4.4. *(Nazarov's UP for the Fourier transform [23]) There exists a constant K, such that for finite Lebesgue measurable sets S, $E \subset \mathbb{R}^n$ and for every $f \in L^2(\mathbb{R}^n)$, we have*

$$Ke^{K(S,E)}\left(\int_{\mathbb{R}^n\setminus S}|f(x)|^2dx+\int_{\mathbb{R}^n\setminus E}|\mathcal{F}[f](w)|^2dw\right)\geq\int_{\mathbb{R}^n}|f(x)|^2dx,$$

where $K(S,E) = Kmin(|S||E|, |S|^{1/n}, \mu(E), \mu(S)|E|^{1/n})$, $\mu(S)$ is the mean width of S and $|S|$ denotes the Lebesgue measure of S.

Now we shall establish Nazarov's UP to the short-time non-sperable linear canonical transform domain.

Theorem 4.8. *(Nazarov's UP) Let $\mathcal{V}_\phi^{\mathbf{M}}[f]$ be the short-time FMT, then under the assumptions of Lemma 4.4 for every $f \in L^2(\mathbb{R}^n)$, the following inequality holds:*

$$\begin{aligned}&||\phi||^2_{L^2(\mathbb{R}^n)}\int_{\mathbb{R}^n}|f(x)|^2dx\\&\leq Ke^{K(S,E)}\left(||\phi||^2_{L^2(\mathbb{R}^n)}\int_{\mathbb{R}^n\setminus S}|f(x)|^2dx+\int_{\mathbb{R}^n}\int_{\mathbb{R}^n(\setminus EB)}|\mathcal{V}_\phi^{\mathbf{M}}[f](w,u)|^2dwdu\right).\end{aligned}$$

Proof. Applying Lemma 4.4 to the function $F(x) \in L^2(\mathbb{R}^n)$ defined in Lemma 3.2, we have

$$\int_{\mathbb{R}^n}|F(x)|^2dx\leq Ke^{K(S,E)}\left(\int_{\mathbb{R}^n\setminus S}|F(x)|^2dx+\int_{\mathbb{R}^n\setminus E}|\mathcal{F}[F](w)|^2dw\right).$$

Now with the help of Lemma 3.7, we have

$$\begin{aligned}&\int_{\mathbb{R}^n}|f(x)\overline{\phi(x-u)}|^2dx\\&\leq Ke^{K(S,E)}\left(\int_{\mathbb{R}^n\setminus S}|f(x)\overline{\phi(x-u)}|^2dx+\int_{\mathbb{R}^n\setminus E}|\sqrt{|det(B)|}\mathcal{V}_\phi^{\mathbf{M}}[f](Bw,u)|^2dw\right).\end{aligned}$$

Integrating both sides with respect to u, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|f(x)\overline{\phi(x-u)}|^2dxdu\\&\leq Ke^{K(S,E)}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n\setminus S}|f(x)\overline{\phi(x-u)}|^2dx+\int_{\mathbb{R}^n\setminus E}|\sqrt{|det(B)|}\mathcal{V}_\phi^{\mathbf{M}}[f](Bw,u)|^2dw\right)du.\end{aligned}$$

Implementing Fubini's theorem, we have

$$||\phi||^2_{L^2(\mathbb{R}^n)}\int_{\mathbb{R}^n}|f(x)|^2dx$$

$$\leq Ke^{K(S,E)}\left(||\phi||^2_{L^2(\mathbb{R}^n)}\int_{\mathbb{R}^n\backslash S}|f(x)|^2dx+|det(B)|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n\backslash E}|\mathcal{V}^{\mathbf{M}}_{\phi}[f](Bw,u)|^2dwdu\right)$$
$$= Ke^{K(S,E)}\left(||\phi||^2_{L^2(\mathbb{R}^n)}\int_{\mathbb{R}^n\backslash S}|f(x)|^2dx+\int_{\mathbb{R}^n}\int_{\mathbb{R}^n(\backslash EB)}|\mathcal{V}^{\mathbf{M}}_{\phi}[f](w,u)|^2dwdu\right),$$

which completes the proof. □

Alternative proof. For any function $f \in L^2(\mathbb{R}^n)$ and a pair of finite measurable subsets S and E of $\mathbb{R}^n$, Nazarov's uncertainty principle in the linear canonical domain reads [32]

$$Ke^{K(S,E)}\left(\int_{\mathbb{R}^n\backslash S}|f(x)|^2dx+\int_{\mathbb{R}^n\backslash EB}|\mathcal{L}_{\mathbf{M}}[f](w)|^2dw\right)\geq\int_{\mathbb{R}^n}|f(x)|^2dx, \tag{4.14}$$

where $K(S,E) = Kmin(|S||E|, |S|^{1/n}, \mu(E), \mu(S)|E|^{1/n})$, $\mu(.)$ is the mean width of measurable subset, and $|.|$ denotes the Lebesgue measure. Moreover, the relationship between the ST-FMT and the FMT is given by

$$\mathcal{V}^{\mathbf{M}}_{\phi}[f](w,u) = \mathcal{L}_{\mathbf{M}}\{f(x)\overline{\phi(x-u)}\}(w). \tag{4.15}$$

Since $f(x)\overline{\phi(x-u)} \in L^2(\mathbb{R}^n)$ then by virtue of (4.14), we have

$$\int_{\mathbb{R}^n}|f(x)\overline{\phi(x-u)}|^2dx$$
$$\leq Ke^{K(S,E)}\left(\int_{\mathbb{R}^n\backslash S}|f(x)\overline{\phi(x-u)}|^2dx+\int_{\mathbb{R}^n\backslash EB}|\mathcal{L}_{\mathbf{M}}[f(x)\overline{\phi(x-u)}](w)|^2dw\right).$$

Using (4.15), we have

$$\int_{\mathbb{R}^n}|f(x)\overline{\phi(x-u)}|^2dx$$
$$\leq Ke^{K(S,E)}\left(\int_{\mathbb{R}^n\backslash S}|f(x)\overline{\phi(x-u)}|^2dx+\int_{\mathbb{R}^n\backslash EB}|\mathcal{V}^{\mathbf{M}}_{\phi}[f](w,u)|^2dw\right).$$

Integrating both sides with respect to u, we obtain

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|f(x)\overline{\phi(x-u)}|^2dxdu$$
$$\leq Ke^{K(S,E)}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n\backslash S}|f(x)\overline{\phi(x-u)}|^2dx+\int_{\mathbb{R}^n\backslash EB}|\mathcal{V}^{\mathbf{M}}_{\phi}[f](w,u)|^2dw\right)du.$$

On implementing the well known Fubini theorem in above equation, we obtain the desired result as

$$||\phi||^2_{L^2(\mathbb{R}^n)}\int_{\mathbb{R}^n}|f(x)|^2dx$$
$$\leq Ke^{K(S,E)}\left(||\phi||^2_{L^2(\mathbb{R}^n)}\int_{\mathbb{R}^n\backslash S}|f(x)|^2dx+\int_{\mathbb{R}^n}\int_{\mathbb{R}^n\backslash EB}|\mathcal{V}^{\mathbf{M}}_{\phi}[f](w,u)|^2dwdu\right).$$

This completes the proof.

5. Numerical example and potential applications

Using simulation, we will offer an illustrated case in this section to demonstrate the proposed ST-FMT (3.1) and some potential applications.

We start with an illustrative example as:

Example 5.1. Consider a function given by $f(x) = e^{-x_1^2-x_2^2}$ and taking the window function $\phi(x) = e^{-\left(\frac{x_1^2}{2}+\frac{x_2^2}{2}\right)}$, which is a well known 2D-Gaussian window function. Then

$$\phi(x-u) = e^{\left(-\frac{(x_1^2+u^2{}_1-2x_1u_1)}{2}-\frac{(x_2^2+u_2^2-2x_2u_2)}{2}\right)}. \tag{5.1}$$

Subsequently, the ST-FMT (3.1) of $f(x)$ with respect to the 2D-Gaussian window function $\phi(x)$, and the real symplectic matrix $\mathbf{M} = (A, B : C, D)$ with

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} \frac{-1}{4} & \frac{-1}{4} \\ \frac{-1}{4} & \frac{1}{4} \end{pmatrix},$$
$$C = \begin{pmatrix} -3 & -6 \\ 3 & 6 \end{pmatrix}, \qquad D = \begin{pmatrix} 2 & 2 \\ \frac{-1}{2} & \frac{1}{2} \end{pmatrix}, \tag{5.2}$$

is given by

$$\begin{aligned}
&\mathcal{V}_\phi^{\mathbf{M}}[f](w,u) \\
&= \frac{1}{(2\pi)\sqrt{|det(B)|}} \int_{\mathbb{R}^2} f(x)\overline{\phi(x-u)} e^{\frac{i(w^T DB^{-1}w + x^T B^{-1}Ax - 2w^T B^{-T}x)}{2}} dx \\
&= \frac{1}{(2\pi)\sqrt{|det(B)|}} \int_{\mathbb{R}^2} e^{-x_1^2-x_2^2} e^{\left(-\frac{(x_1^2+u^2{}_1-2x_1u_1)}{2}-\frac{(x_2^2+u_2^2-2x_2u_2)}{2}\right)} \\
&\qquad \times e^{\frac{i(w^T DB^{-1}w + x^T B^{-1}Ax - 2w^T B^{-T}x)}{2}} dx_1 dx_2.
\end{aligned}$$

Moreover, we have

$$w^T DB^{-1} w = (w_1 \quad w_2)\begin{pmatrix} -8 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = -8w_1^2 + 2w_2^2, \tag{5.3}$$

$$w^T B^{-T} x = (w_1 \quad w_2)\begin{pmatrix} -2 & -2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -2(w_1+w_2)x_1 - 2(w_1-w_2)x_2, \tag{5.4}$$

$$x^T B^{-1} A x = (x_1 \quad x_2)\begin{pmatrix} -4 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -4x_1^2 + 4x_2^2. \tag{5.5}$$

Plugging Eqs (5.3)–(5.5) in Eq (5.3), we get

$$\begin{aligned}
&\mathcal{V}_\phi^{\mathbf{M}}[f](w,u) \\
&= \frac{2\sqrt{2}}{2\pi} \int_{\mathbb{R}^2} e^{-x_1^2-x_2^2} e^{-\left(\frac{x_1^2+u_1^2-2x_1u_1}{2}\right)-\left(\frac{x_2^2+u_2^2-2x_2u_2}{2}\right)}
\end{aligned}$$

$$\times e^{\frac{i}{2}(-8w_1^2+2w_2^2+4w_1x_1+4w_2x_1+4w_1x_2-4w_2x_2-4x_1^2+4x_2^2)}dx_1dx_2$$

$$= \frac{\sqrt{2}e^{-\left(\frac{u_1^2+u_2^2}{2}\right)}}{\pi} e^{i\left(-4w_1^2+w_2^2\right)} \int_{\mathbb{R}} e^{-\left(\frac{3}{2}+2i\right)x_1^2+(2i(w_1+w_2)+u_1)x_1} dx_1$$

$$\times \int_{\mathbb{R}} e^{-\left(\frac{3}{2}-2i\right)x_2^2+(2i(w_1-w_2)+u_2)x_2} dx_2$$

$$= \frac{2\sqrt{2}}{5} e^{-\left((4w_1^2-w_2^2)i+\frac{u_1^2+u_2^2}{2}\right)} e^{\left(\frac{(u_1+2i(w_1+w_2))^2}{6+4i}+\frac{(u_2+2i(w_1-w_2))^2}{6-4i}\right)}. \tag{5.6}$$

It is clear from (5.6) that the ST-FMT given in Definition 3.1 can be visualized as a signal with four parameters w_1, w_2, u_1 and u_2. A simultaneous visualization of the ST-FMT in all the four variables is hardly possible. Therefore, in order to have a sound visualization, some variables have either to be fixed or eliminated, so that one is restricted to a particular section of the parameter space. This gives rise to several representations associated with the ST-FMT (3.1).

(i) Position Representation: If $\mathcal{V}_\phi^{\mathbf{M}}[f](w,u)$ is solely regarded as a function of the translation parameter u in (5.6), then the ST-FMT (3.1) yields a position representation system $\mathcal{V}_\phi^{\mathbf{M}}[f](.,u)$, which is widely used in image processing: Shape and contours of objects, pattern recognition, detection of position, image filtering and so on. The real and imaginary parts of the position representation of $\mathcal{V}_\phi^{\mathbf{M}}[f](w,u)$ for the symplectic matrix $\mathbf{M}$ given by (5.2) are presented in Figure 1.

(ii) Frequency Representation: If in $\mathcal{V}_\phi^{\mathbf{M}}[f](w,u)$ the translation parameter u is kept constant, then the ST-FMT (3.1) is regarded as a function of frequency w. The frequency representation system $\mathcal{V}_\phi^{\mathbf{M}}[f](w,.)$ is especially useful in situations where frequency behaviour is important. The real and imaginary parts of the frequency representation of $\mathcal{V}_\phi^{\mathbf{M}}[f](w,u)$ for the symplectic matrix $\mathbf{M}$ given by (5.2) and $u=(0,0)$ is presented in Figure 2.

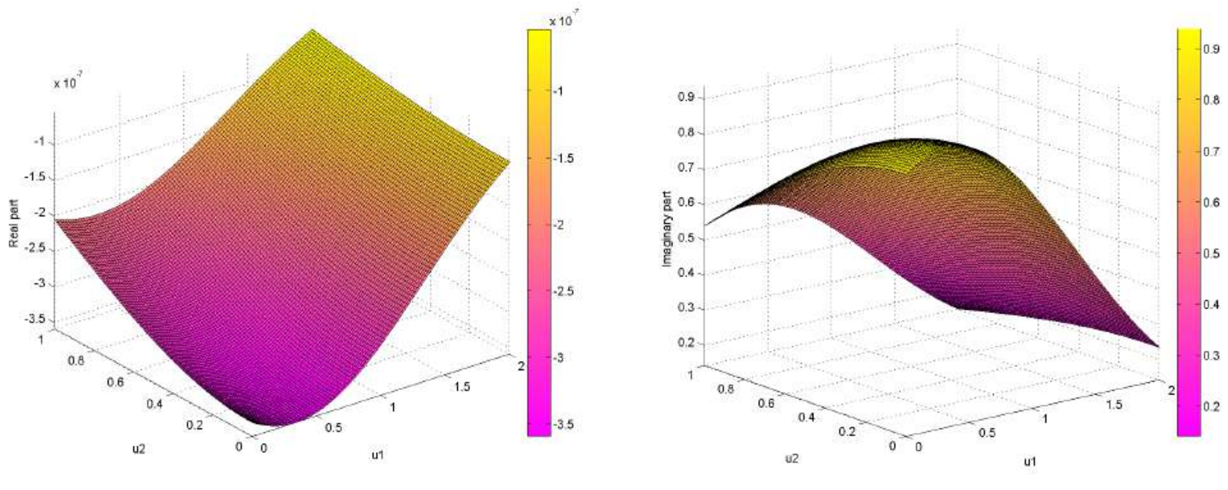

Figure 1. Real and imaginary parts of the position of an Example 5.1 for $w=(2,1)$.

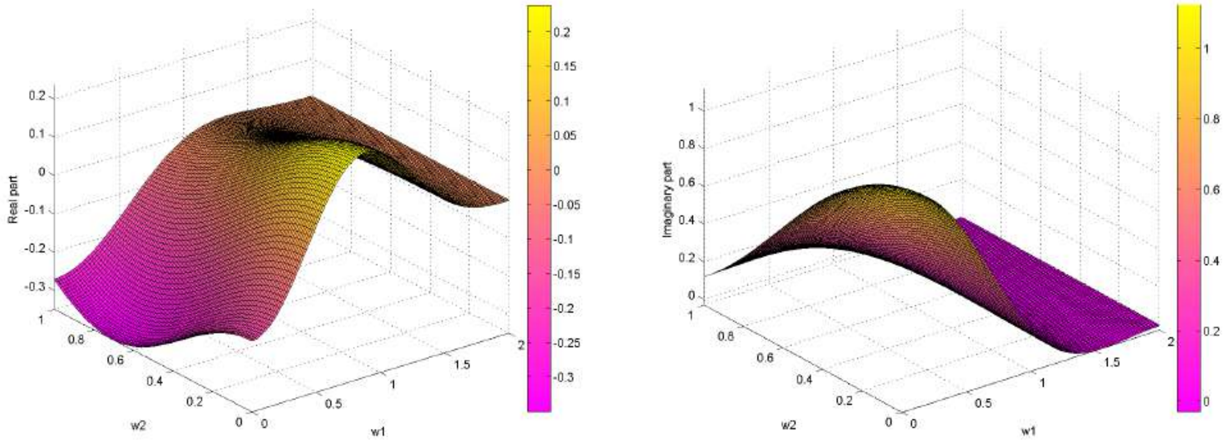

Figure 2. Real and imaginary parts of the frequency of an Example 5.1 for $t = (0, 0)$.

The UPs in the ST-FMT environment share the same philosophy as the UPs in the FMT environment. The ST-FMT case is subject to suitably incorporate the window function. The purpose of this section is to demonstrate the value and significance of the theorems in signal processing through a few hypothetical applications. The band-widths are frequently estimated using UPs. For example, if the spread of the signal in the time domain ($T_{\mathbf{M}}$) is known, then the bandwidth of a system that performs ST-FMT cannot be narrower than $\frac{n^2\sigma^2_{min}(B)}{16T^2_{\mathbf{M}}\pi^2}$. Thus, the derived Theorem 4.3 can be used in the estimation of effective bandwidths in the ST-FMT. Other potential applications can be found in the estimation of the lower bound of an integral as if a signal in Theorem 4.3 is determined, it is difficult to obtain the value of the left side of Theorem 4.3 by calculating the integral. However, an estimation of the left side of Theorem 4.3 can be easily obtained based on Theorem 4.3. The UPs have some applications in signal recovery. Donoho and Stark [16] studied the problem of signal recovery. The potential applications can be found in signal modulations and filter design, such as the methods presented in [39, 40]. Authors in [8] have shown that the uncertainty principle related to the spread in the FMT domain has an immediate application in the discussion of some well-known optical propagation models, such as wave propagation through an aperture, free-space propagation and pulse propagation in optical fibres. The correlative result has been applied to the ST-FMT domain as well.

6. Conclusions

In this paper we presented a novel concept of short-time free metaplectic transform. Based on the properties of ST-FMT and FMT, the relationship between these two notations are presented. Important properties such as boundedness, reconstruction formula and Moyals formula are derived. Then, we extend some different uncertainty principles from quantum mechanics including Lieb's inequality, Pitt's UP, Heisenberg's UP, Hausdorff-Young inequality, Hardy's UP, Beurling's UP, Logarithmic UP and Nazarov's UP which have already been well studied in the ST-FMT domain. Finally, we provided a numerical example and some potential applications of the proposed ST-FMT.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

References

1. G. B. Folland, *Harmonic analysis in phase space*, Annals of Mathematics Studies, Princeton: Princeton University Press, **120** (1989). https://doi.org/10.1515/9781400882427
2. Z. C. Zhang, Unified Wigner-Ville distribution and ambiguity function in the linear canonical transform domain, *Signal Process.*, **114** (2015), 45–60. https://doi.org/10.1016/j.sigpro.2015.02.016
3. Z. C. Zhang, New Wigner distribution and ambiguity function based on the generalized translation in the linear canonical transform domain, *Signal Process.*, **118** (2016), 51–61. https://doi.org/10.1016/j.sigpro.2015.06.010
4. Z. C. Zhang, T. Yu, M. K. Luo, K. Deng, Estimating instantaneous frequency based on phase derivative and linear canonical transform with optimized computational speed, *IET Signal Process.*, **12** (2018), 574–580. https://doi.org/10.1049/iet-spr.2017.0469
5. Z. C. Zhang, J. Shi, X. P. Liu, L. He, M. Han, Q. Z. Li, et al., Sampling and reconstruction in arbitrary measurement and approximation spaces associated with linear canonical transform, *IEEE Trans. Signal Process.*, **64** (2016), 6379–6391. https://doi.org/10.1109/TSP.2016.2602808
6. M. Moshinsky, C. Quesne, Linear canonical transformations and their unitary representations, *J. Math. Phys.*, **12** (1971), 1772–1780. https://doi.org/10.1063/1.1665805
7. F. A. Shah, A. Y. Tantary, Lattice-based multi-channel sampling theorem for linear canonical transform, *Digit Signal Process.*, **117** (2021), 103168. https://doi.org/10.1016/j.dsp.2021.103168
8. Z. Zhang, Uncertainty principle of complex-valued functions in specific free metaplectic transformation domains, *J. Fourier Anal. Appl.*, **27** (2021). http://dx.doi.org/10.1007/s00041-021-09867-6
9. Z. Zhang, Uncertainty principle for real functions in free metaplectic transformation domains, *J. Fourier Anal. Appl.*, **25** (2019), 2899–2922. https://doi.org/10.1007/s00041-019-09686-w
10. M. Y. Bhat, A. H. Dar, Convolution and correlation theorems for Wigner-Ville distribution associated with the quaternion offset linear canonical transform, *Signal Image Video P.*, **16** (2022), 1235–1242. https://doi.org/10.1007/s11760-021-02074-2

11. M. Y. Bhat, A. H. Dar, Wavelet packets associated with linear canonical transform on spectrum, *Int. J. Wavelets Multi.*, **19** (2021). https://doi.org/10.1142/S0219691321500302

12. M. Y. Bhat, A. H. Dar, Fractional vector-valued nonuniform MRA and associated wavelet packets on $L^2(\mathbb{R}^2, C^M)$, *Fract. Calc. Appl. Anal.*, **25** (2022), 687–719. https://doi.org/10.1007/s13540-022-00035-1

13. M. Y. Bhat, A. H. Dar, The algebra of 2D Gabor quaternionic offset linear canonical transform and uncertainty principles, *J. Anal.*, **30** (2022), 637–649. https://doi.org/10.1007/s41478-021-00364-z

14. M. Y. Bhat, A. H. Dar, Donoho starks and Hardys uncertainty principles for the short-time quaternion offset linear canonical transform, *Filomat*, **37** (2023), 4467– 4480.

15. W. Heisenberg, Uber den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik, *Zeitschrift Physik*, **43** (1927), 172–198. https://doi.org/10.1007/BF01397280

16. D. L. Donoho, P. B. Stark, Uncertainty principles and signal recovery, *SIAM J. Appl. Math.*, **49** (1989), 906–931. https://doi.org/10.1137/0149053

17. B. Ricaud, B. A. Torrésani, Survey of uncertainty principles and some signal processing applications, *Adv. Comput. Math.*, **40** (2014), 629–650. https://doi.org/10.1007/s10444-013-9323-2

18. Q. Zhang, Zak transform and uncertainty principles associated with the linear canonical transform, *IET Signal Process.*, **10** (2016), 791–797. https://doi.org/10.1049/iet-spr.2015.0514

19. H. Y. Huo, Uncertainty principles for the offset linear canonical transform, *Circ. Syst. Signal Process.*, **38** (2019), 395–406. https://doi.org/10.1007/s00034-018-0863-z

20. Z. C. Zhang, N-dimensional Heisenberg's uncertainty principle for fractional Fourier transform, *Phys. Eng. Math.*, 2019. Available from: `https://arxiv.org/abs/1906.05451`.

21. Y. G. Li, B. Z. Li, H. F. Sun, Uncertainty principles for Wigner-Ville distribution associated with the linear canonical transforms, *Abstr. Appl. Anal.*, **2014** (2014), 1–9. https://doi.org/10.1155/2014/470459

22. F. L. Nazarov, Local estimates for exponential polynomials and their applications to inequalities of the uncertainty principle type, *Algebra I Anal.* **5** (1993), 663–717.

23. P. Jaming, Nazarov's uncertainty principle in highter dimension, *J. Approx. Theory*, **149** (2007), 30–41. https://doi.org/10.1016/j.jat.2007.04.005

24. A. Bonami, B. Demange, P. Jaming, Hermite functions and uncertainty principles for the Fourier and the windowed Fourier transforms, *Rev. Mat. Iberoam.*, **19** (2003), 23–55. https://doi.org/10.4171/RMI/337

25. S. C. Bagchi, S. K. Ray, Uncertainty principles like Hardy's theorem on some Lie groups, *J. Aust. Math. Soc.*, 1999, 289–302. https://doi.org/10.1017/S1446788700035886

26. W. Beckner, Pitt's inequality and the uncertainty principle, *Proc. Am. Math. Soc.*, **123** (1995), 1897–1905. https://doi.org/10.1090/S0002-9939-1995-1254832-9

27. G. B. Folland, A. Sitaram, The uncertainty principle: A mathematical survey, *J. Fourier Anal. Appl.*, **3** (1997), 207–238. https://doi.org/10.1007/BF02649110

28. M. Y. Bhat, A. H. Dar, Uncertainty principles for quaternion linear canonical S-transform, *Funct. Anal.*, 2021.

29. M. Y. Bhat, A. H. Dar, Uncertainty inequalities for 3D octonionic-valued signals associated with octonion offset linear canonical transform, *Signal Process.*, 2021.

30. M. Y. Bhat, Windowed linear canonical transform: Its relation to windowed Fourier transform and uncertainty principles, *J. Inequal. Appl.*, **4** (2022). https://doi.org/10.1186/s13660-021-02737-1

31. M. Y. Bhat, A. H. Dar, Octonion spectrum of 3D short-time LCT signals, *Signal Process.*, 2021. https://doi.org/10.1016/j.ijleo.2022.169156

32. R. Jing, B. Liu, R. Li, R. Liu, The N-dimensional uncertainty principle for the free metaplectic transformation, *Mathematics*, **8** (2020). http://dx.doi.org/ 10.3390/math8101685

33. H. M. Srivastava, F. A. Shah, W. Z. Lone, Non-separable linear canonical wavelet transform, *Symmetry*, **13** (2021), 2182. https://doi.org/10.3390/ sym13112182

34. M. Bahri, R. Ashino, Some properties of windowed linear canonical transform and its logarithmic uncertainty principle, *Int. J. Wavelets Multi.*, **14** (2016), 1650015. https://doi.org/10.1142/S0219691316500156

35. F. A. Shah, A. Y. Tantary, Multi-dimensional linear canonical transform with applications to sampling andmultiplicative filtering, *Multidim. Syst. Sign. P.*, **117** (2021) 103168. https://doi.org/10.1007/s11045-021-00816-6

36. M. Bahri, Windowed linear canonical transform: Its relation to windowed Fourier transform and uncertainty principles, *J. Inequal. Appl.*, **2022** (2022). https://doi.org/10.1186/s13660-021-02737-1

37. G. H. Hardy, A theorem concerning Fourier transforms, *J. Lond. Math. Soc.*, **8** (1933), 227–231. https://doi.org/10.1112/jlms/s1-8.3.227

38. L. Escauriaza, C. E. Kenig, G. Ponce, L. Vega, The sharp Hardy uncertainty principle for Schodinger evolutions, *Duke Math. J.*, **155** (2010), 163–187. https://doi.org/10.1215/00127094-2010-053

39. Y. J. Cao, B. Z. Li, Y. G. Li, Y. H. Chen, Logarithmic uncertainty relations for odd or even signals associate with Wigner-Ville distribution, *Circ. Syst. Signal Pr.*, **35** (2016), 2471–2486. https://doi.org/10.1007/s00034-015-0146-x

40. R. Skoog, An uncertainty principle for functions vanishing on a half-line, *IEEE T. Circ. Theory*, **173** (1970), 241–243. https://doi.org/10.1109/TCT.1970.1083079

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