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TIGHT AFFINE, QUASI-AFFINE WAVELET FRAMES ON LOCAL FIELDS OF POSITIVE CHARACTERISTIC

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Abstract

Frames have very important and interesting properties which make them very useful in the characterization of function spaces, signal and image processing, sampling theory and many other fields. An important tool for the construction of tight wavelet frames on local fields of positive characteristic is the tool of unitary extension principle. In this paper, we continue the study based on the extension principles and give an explicit construction of a class of tight affine frames as well as quasi-affine wavelet frames on local fields of positive characteristic.

1. Introduction

The frame was first introduced by Duffin and Schaeffer [4] in the study of non-harmonic Fourier series in 1952, reintroduced in 1986 by Daubechies et al., and popularized from then on. Frames and their duals have very important and interesting properties which make them very useful in the

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characterization of function spaces, signal and image processing, sampling theory and many other fields. A frame is a family of elements in a separable Hilbert space which allows stable not necessarily unique decomposition of arbitrary elements into expansions of frame elements. A sequence $\{f_k\}_{k=1}^{\infty}$ of elements of a Hilbert space $\mathbb{H}$ is called a *frame* for $\mathbb{H}$ if there exist constants $A, B > 0$ such that for all $f \in \mathbb{H}$,

$$A\| f \|_2^2 \le \sum_{k=1}^{\infty} | \langle f, f_k \rangle |^2 \le B\| f \|_2^2,$$

where A and B are the lower and upper frame bounds, respectively. If $A = B$, then the frame is called a *tight frame*. If $A = B = 1$, then the frame is called a *normalized tight frame*. The particular frames of interest to us will be the frames in the space $L^2(\mathbb{R})$ which are generated by the combined action of dilations and translations of finite number of functions. In order to describe these frames, for $a, b \in \mathbb{R}$ with $a > 1$ and $b > 0$, we define wavelet systems as

$$\mathcal{F}(\psi, a, b) = \{\psi_{j,k} =: a^{j/2}\psi(a^j x - kb) : j, k \in \mathbb{Z}\}.$$

Wavelet systems $\mathcal{F}(\psi, a, b)$ that form frames for $L^2(\mathbb{R})$ have a wide variety of applications [3, 7]. Therefore, one of the fundamental problems in the study of wavelet frames is to find conditions on ψ, a and b such that the system $\mathcal{F}(\psi, a, b)$ forms a frame. In 1990, Daubechies obtained the first result on the necessary conditions for wavelet frames, and then in 1993, Chui and Shi obtained an improved result. Cassaza and Christensen provided a stronger version of Daubechies's sufficient condition for wavelet frames in $L^2(\mathbb{R})$. In recent years, these conditions have been further improved and investigated by many authors [15, 17, 20]. All these concepts are developed on regular lattices, that is the translation set is always a group. Gabardo et al. in [5, 6] have developed the concept of nonuniform wavelets on $L^2(\mathbb{R})$. Here, the translation set is not a discrete subgroup of $\mathbb{R}$, but a union of two lattices. Subsequently, nonuniform wavelet frames associated with spectral pairs were constructed by Shah and Bhat [16] using the machinery of Fourier transforms. In fact, they obtained necessary and sufficient conditions for the nonuniform wavelet system to be a frame for $L^2(K)$. Recent results in this direction can also be found in [7, 9-17] and the references therein. Separable and non-separable wavelet frames have been widely studied by Pan and Wang [12]. Here, the authors have shown that the tree-structured wavelet decomposition based on non-separable wavelet frames has better performance than that based on separable ones.

Benedetto and Benedetto [1] constructed wavelets on some specific groups $\mathbb{G}$ like *p*-adic rational group $\mathbb{Q}_p$, Cantor dyadic group $\mathbb{F}^2((t))$, etc. You et al. [20] have provided a construction of new nontensor product of wavelet filter banks for providing a blind watermarking scheme. Here, they have utilised special symmetric matrices in the construction of these nontensor product of wavelet filter banks that can capture the singularities in all the directions. Here, in this paper, we concentrate on local fields. The local fields have been deeply studied in [2, 15-18].

We turn to investigate tight affine, quasi-affine wavelet frames on local fields of positive characteristic. The paper is organized as follows. Section 2 briefly introduces some notations of local fields needed throughout the paper. Section 3 is devoted to the discussion of tight affine, quasi-affine wavelet frames on local fields. Finally, Section 4 concludes the paper.

2. Preliminaries on Local Fields

A field K equipped with a topology is called a *local field* if both the additive K^+ and multiplicative groups K^* of K are locally compact Abelian groups. The local fields are essentially of two types: zero and positive characteristic (excluding the connected local fields $\mathbb{R}$ and $\mathbb{C}$). Examples of local fields of characteristic zero include the *p*-adic field $\mathbb{Q}_p$, whereas local fields of positive characteristic are the Cantor dyadic group and the Vilenkin

p-groups. Even though the structures and metrics of local fields of zero and positive characteristics are similar, their wavelet and multiresolution analysis theory are quite different. In recent years, local fields have attracted the attention of several mathematicians, and have found innumerable applications not only to number theory but also to representation theory, division algebras, quadratic forms and algebraic geometry. As a result, local fields are now consolidated as a part of the standard repertoire of contemporary mathematics. For more about local fields and their applications, we refer to the monographs [13, 19].

Let K be a field and a topological space. Then K is called a *local field* if both K^+ and K^* are locally compact Abelian groups, where K^+ and K^* denote the additive and multiplicative groups of K, respectively. If K is any field and is endowed with the discrete topology, then K is a local field. Further, if K is connected, then K is either $\mathbb{R}$ or $\mathbb{C}$. If K is not connected, then it is totally disconnected. Hence, by a local field, we mean a field K which is locally compact, non-discrete and totally disconnected. The p-adic fields are examples of local fields. More details are referred to [13, 19]. In the rest of this paper, we use the symbols $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{Z}$ to denote the sets of natural, non-negative integers and integers, respectively.

The Fourier transform $\hat{f}$ of a function $f \in L^1(K) \cap L^2(K)$ is defined by

$$\hat{f}(\xi) = \int_K f(x)\overline{\chi_\xi(x)}dx. \tag{2.1}$$

It is noted that

$$\hat{f}(\xi) = \int_K f(x)\overline{\chi_\xi(x)}dx = \int_K f(x)\chi(-\xi x)dx.$$

Furthermore, the properties of Fourier transform on local field K are much similar to those of on the real line. In particular, Fourier transform is unitary on $L^2(K)$.

We now impose a natural order on the sequence $\{u(n)\}_{n=0}^{\infty}$. We have $\mathfrak{D}/\mathfrak{B} \cong GF(q)$, where $GF(q)$ is a c-dimensional vector space over the field $GF(p)$. We choose a set $\{1 = \zeta_0, \zeta_1, \zeta_2, ..., \zeta_{c-1}\} \subset \mathfrak{D}^*$ such that $\operatorname{span}\{\zeta_j\}_{j=0}^{c-1} \cong GF(q)$. For $n \in \mathbb{N}_0$ satisfying

$$0 \le n < q, \quad n = a_0 + a_1 p + \cdots + a_{c-1}p^{c-1}, \quad 0 \le a_k < p, \text{ and}$$
$$k = 0, 1, ..., c-1,$$

we define

$$u(n) = (a_0 + a_1\zeta_1 + \cdots + a_{c-1}\zeta_{c-1})\mathfrak{p}^{-1}. \tag{2.2}$$

Also, for $n = b_0 + b_1 q + b_2 q^2 + \cdots + b_s q^s$, $n \in \mathbb{N}_0$, $0 \le b_k < q$, $k = 0, 1, 2, ..., s$, we set

$$u(n) = u(b_0) + u(b_1)\mathfrak{p}^{-1} + \cdots + u(b_s)\mathfrak{p}^{-s}. \tag{2.3}$$

Hence, $u(n)$ for all $n \in \mathbb{N}_0$ is defined. Generally, we cannot say that $u(m+n) = u(m) + u(n)$. But, if $r, k \in \mathbb{N}_0$ and $0 \le s < q^k$, then $u(rq^k + s) = u(r)\mathfrak{p}^{-k} + u(s)$. Moreover, we can verify that $u(n) = 0$ if and only if $n = 0$ and $\{u(\ell) + u(k) : k \in \mathbb{N}_0\} = \{u(k) : k \in \mathbb{N}_0\}$ for a fixed $\ell \in \mathbb{N}_0$. From hereafter, we will use $\chi_n = \chi_{u(n)}$, $n \ge 0$.

Consider the local field K to be of positive characteristic t and let $\zeta_0, \zeta_1, \zeta_2, ..., \zeta_{c-1}$ be as above. Then the character χ on K is defined as follows:

$$\chi(\zeta_\mu \mathfrak{p}^{-j}) = \begin{cases} \exp(2\pi i/t), & \mu = 0 \text{ and } j = 1, \\ 1, & \mu = 1, ..., c-1 \text{ or } j \ne 1. \end{cases} \tag{2.4}$$

3. Affine and Quasi-affine Tight Frames

For $j \in \mathbb{Z}$ and $k \in \mathbb{N}_0$, we define the dilation operator D_j and the translation operator T_k as

$$D_j f(\cdot) = q^{j/2} f(\mathfrak{p}^{-j} \cdot) \text{ and } T_k f(\cdot) = f(\cdot - k),\ f \in L^2(K).$$

Given $j \in \mathbb{Z}$, we have $T_{u(k)} D_j = D_j T_{\mathfrak{p}^{-j} u(k)}$. Moreover, a space V is said to be *invariant* under integer-shift if for any function $f \in V$, $T_j f \in V$ for $j \in \mathbb{Z}$.

A system $X \subset L^2(K)$ is said to be a *tight frame* for $L^2(K)$ if for any $f \in L^2(K)$,

$$\| f \|^2_{L^2(K)} = \sum_{g \in X} |\langle f, g\rangle|^2$$

holds. This can be rewritten as

$$f = \sum_{g \in X} \langle f, g\rangle g, \quad f \in L^2(K).$$

Definition 3.1. Let $\Psi := \{\psi^1, \psi^2, ..., \psi^L\}$ be a finite family of functions in $L^2(K)$. Then the *affine system* generated by Ψ is the collection

$$X(\Psi) := \{\psi^\ell_{j,k} : 1 \le \ell \le L;\ j \in \mathbb{Z},\ k \in \mathbb{N}_0\},$$

where $\psi^\ell_{j,k} = q^{j/2} \psi^\ell(\mathfrak{p}^{-j} \cdot - u(k)) = D_j T_{u(k)} \psi^\ell$.

The wavelet symbols $\psi^1, \psi^2, ..., \psi^L$ are called the *orthonormal wavelets* whenever $X(\Psi)$ forms an orthonormal basis of $L^2(K)$. These symbols are called the *tight framelets* if the system $X(\Psi)$ forms a tight frame for $L^2(K)$.

The construction of tight framelets often starts with the construction of MRA, which is built on refinable functions. A compactly supported function

$\varphi \in L^2(K)$ is called *refinable* if it satisfies a refinement equation:

$$\varphi(x) = q^{1/2} \sum_{k \in \mathbb{N}_0} h_k \varphi(\mathfrak{p}^{-1}x - u(k)) \tag{3.1}$$

for some $\{h_k : k \in \mathbb{N}_0\} \in l^2(\mathbb{N}_0)$. The Fourier transform of (3.1) yields

$$\hat{\varphi}(\xi) = m_0(\mathfrak{p}\xi)\hat{\varphi}(\mathfrak{p}\xi), \tag{3.2}$$

where

$$m_0(\xi) = q^{-1/2} \sum_{k \in \mathbb{N}_0} h_k \overline{\chi_k(\xi)}$$

is an integral periodic function in $L^2(\mathfrak{D})$ and is often called the *refinement symbol* of φ. Given a refinable function $\varphi \in L^2(K)$ with $\hat{\varphi}(0) \neq 0$, the sequence of subspaces $\{V_j\}_{j \in \mathbb{Z}}$ defined by

$$V_j = \overline{\operatorname{span}}\{\varphi(\mathfrak{p}^{-j} - u(k)) : k \in \mathbb{N}_0\},$$

will form an MRA for $L^2(K)$. Recall that $\{V_j : j \in \mathbb{Z}\}$ is called an *MRA* if it satisfies: (i) $V_j \subset V_{j+1}$ for every $j \in \mathbb{Z}$; (ii) $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(K)$ and (iii) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$. In this paper, we only consider the refinable function $\varphi \in L^2(K)$ satisfying the following properties:

$$\lim_{j \to \infty} \hat{\varphi}(\mathfrak{p}^j \xi) = 1 \text{ for a.e. } \xi \in \mathfrak{D}$$

and

$$\sum_{k \in \mathbb{N}_0} |\hat{\varphi}(\xi + u(k))|^2 = 1 \text{ for a.e. } \xi \in \mathfrak{D}.$$

Given an MRA generated by the refinable function φ, one can construct (see [4]) a set of MRA-based tight framelets $\Psi := \{\psi^1, \psi^2, \ldots, \psi^L\} \subset V_1$ which is defined by

$$\hat{\psi}^{\ell}(\xi) = m_{\ell}(\mathfrak{p}\xi)\hat{\varphi}(\mathfrak{p}\xi), \tag{3.3}$$

where

$$m_{\ell}(\xi) = q^{-1/2} \sum_{k \in \mathbb{N}_0} h_k^{\ell} \overline{\chi_k(\xi)}, \quad 1 \le \ell \le L$$

are the integral periodic functions in $L^2(\mathfrak{D})$ and called the *framelet symbols* or *wavelet masks*. The UEP gives conditions on $m_0, m_1, ..., m_L$ such that Ψ becomes a set of tight framelets with $X(\Psi)$ being a tight frame for $L^2(K)$.

Theorem 3.2 (Unitary Extension Principle)**.** *Let* $\varphi \in L^2(K)$ *be such that* $\{\varphi(\cdot\, u(k)) : k \in \mathbb{N}_0\}$ *is an orthonormal system. Let* $V = span\{q^{1/2}\varphi(\mathfrak{p}^1 \cdot u(k)) : k \in \mathbb{N}_0\}$. *Let* $m_l = q^{1/2}\sum_{k=0}^{l} h_k^l \overline{\chi_k(\xi)}, \quad 0 \le l \le q,$ *where* $\{h_k^l, k \in \mathbb{N}_0\} \in \ell^2(\mathbb{N}_0)$ *for* $0 \le l \le q$. *Define* $\hat{\psi}_l(\xi) = m_l(\mathfrak{p}\xi)\hat{\varphi}_l(\mathfrak{p}\xi)$. *Then* $\{\psi_l(\cdot\, u(k)) : 0 \le l \le q, k \in \mathbb{N}_0\}$ *is an orthonormal system in V if and only if the matrix*

$$M(\xi) = (m_l(\mathfrak{p}\xi + \mathfrak{p}u(k)))_{l,k=0}^{q-1}$$

is unitary for a.e. $\xi \in \mathfrak{D}$. *Moreover*, $\{\psi_l(\cdot\, u(k)) : 0 \le l \le q, k \in \mathbb{N}_0\}$ *is an orthonormal basis of V whenever it is orthonormal*.

The deconvolution process has to be formulated by quasi-affine systems. We define the quasi-affine systems from a fixed level say J on local fields of positive characteristic.

Definition 3.3. The *quasi-affine system* from level J, generated by Ψ, is defined as

$$\widetilde{X}_J(\Psi) := \{\psi_{j,k}^{\ell} : 1 \le \ell \le L;\ j \in \mathbb{Z}, k \in \mathbb{N}_0\},$$

where

$$\widetilde{\psi}^{\ell}_{j,k} = \begin{cases} D_j T_{u(k)} \psi^{\ell} = q^{j/2} \psi^{\ell}(\mathfrak{p}^{-j} \cdot - u(k)), & j \geq J; \\ q^{\frac{j-J}{2}} T_{\mathfrak{p}^J u(k)} D_j \psi^{\ell} = q^{j - \frac{J}{2}} \psi^{\ell}(\mathfrak{p}^{-j}(\cdot - \mathfrak{p}^J u(k))), & j < J. \end{cases}$$

The quasi-affine system can be obtained from oversampling the affine system. This means that we oversample the affine system beginning from the level $J - 1$ and downward to a q^J-shift invariant system. Thus, the whole quasi-affine system is a q^J-shift invariant system. This system from level 0 was first introduced in [2].

In the present paper, we make use of quasi-interpolatory operator. Let φ be the refinable function of the given MRA $\{V_j\}_{j \in \mathbb{Z}}$ and $\Psi := \{\psi^1, \psi^2, \ldots, \psi^L\} \subset V_1$ be the set of corresponding tight framelets obtained via UEP. The quasi-interpolatory operator in the affine system $X(\Psi)$ generated by Ψ is defined as

$$P_j : f(x) \mapsto \sum_{k \in \mathbb{N}_0} \langle f, \varphi_{j,k} \rangle \varphi_{j,k}(x), \quad j \in \mathbb{Z}, \quad f \in L^2(K).$$

It is obvious that $P_j f(x) \in V_j$. Moreover, the quasi-interpolatory operator has the *truncated representation*

$$Q_j : f(x) \mapsto \sum_{\ell=1}^{L} \sum_{k \in \mathbb{N}_0} \langle f, \psi^{\ell}_{u,k} \rangle \psi^{\ell}_{u,k}(x), \quad j > u \in \mathbb{Z}.$$

Furthermore, from the standard framelet decomposition, we have

$$P_{j+1} f(x) = P_j f(x) + \sum_{\ell=1}^{L} \sum_{k \in \mathbb{N}_0} \langle f, \psi^{\ell}_{j,k} \rangle \psi^{\ell}_{j,k}(x) \text{ and}$$

$$P_j f(x) = Q_j f(x). \tag{3.4}$$

On considering MRA based on quasi-affine system $\widetilde{X}_J(\Psi)$ generated by Ψ, the spaces V_j, $j < J$ in the MRA based on affine system will be replaced by $\widetilde{V}_j^J$, $j < J$ for the quasi-affine system. Comparing with the space V_j spanned by functions $\varphi_{j,k}$, the space $\widetilde{V}_j^J$, $j < J$ is spanned by the functions $\widetilde{\varphi}_{j,k}$, where

$$\widetilde{\varphi}_{j,k} = \begin{Bmatrix} D_j T_{u(k)}\varphi = q^{j/2}\psi^{\ell}(\mathfrak{p}^{-j}\cdot - u(k)), & j \geq J \\ q^{\frac{j-J}{2}} T_{\mathfrak{p}^J u(k)} D_j \varphi = q^{j-\frac{J}{2}}\varphi(\mathfrak{p}^{-j}(\cdot - \mathfrak{p}^J u(k))), & j < J \end{Bmatrix}. \quad (3.5)$$

The spaces $\widetilde{V}_j^J$, $j < J$ are q^J-shift invariant. Similar to the affine system, we can define quasi-interpolatory operator $\widetilde{P}_j^J$ and the truncated operator $\widetilde{Q}_j^J$, for the quasi-affine system as

$$\widetilde{P}_j^J : f(x) \mapsto \sum_{k \in \mathbb{N}_0} \langle f, \widetilde{\varphi}_{j,k}\rangle \widetilde{\varphi}_{j,k}(x) \quad (3.6)$$

and

$$\widetilde{Q}_j^J : f(x) \mapsto \sum_{\ell=1}^{L} \sum_{k \in \mathbb{N}_0} \langle f, \widetilde{\psi}_{u,k}^{\ell}\rangle \widetilde{\psi}_{u,k}^{\ell}(x), \quad j > u \in \mathbb{Z}. \quad (3.7)$$

The quasi-interpolatory operator $\widetilde{P}_j^J$ takes f of $L^2(K)$ to $\widetilde{V}_j^J$. From the system (3.5), it is clear that $\widetilde{P}_j^J = P_j$ whenever $j \geq J$ and these operators differ only for the case $j < J$. Furthermore, for any $f \in L^2(K)$ and $j < J$, we have

$$\widetilde{P}_j^J = \sum_{k \in \mathbb{N}_0} \langle f, \widetilde{\varphi}_{j,k}\rangle \widetilde{\varphi}_{j,k}(x)$$

$$= \sum_{k \in \mathbb{N}_0} \langle f, q^{\frac{j-J}{2}} T_{\mathfrak{p}^J u(k)} D_j \varphi\rangle q^{\frac{j-J}{2}} T_{\mathfrak{p}^J u(k)} D_j \varphi(x)$$

$$= q^{j-J-0} \sum_{k \in \mathbb{N}_0} \langle f, D_J D_{-J} T_{\mathfrak{p}^J u(k)} D_j \varphi \rangle D_J D_{-J} T_{\mathfrak{p}^J u(k)} D_j \varphi(x)$$

$$= D_J \sum_{k \in \mathbb{N}_0} \langle D_{-J} f, q^{\frac{j-J-0}{2}} T_{u(k)} D_{j-J} \varphi \rangle q^{\frac{j-J-0}{2}} T_{\mathfrak{p}^J u(k)} D_{j-J} \varphi(x)$$

$$= D_J \widetilde{P}^0_{j-J} D_{-J} f(x).$$

From the above system, it is clear that one needs to understand the case $J = 0$ only. So, we simplify our notations by setting

$$\widetilde{P}_j = \widetilde{P}^0_j, \widetilde{Q}_j = \widetilde{Q}^0_j \text{ and } \widetilde{V}_j = \widetilde{V}^0_j.$$

Thus, it will be sufficient to study the properties for $\widetilde{P}_j$ and the corresponding spaces $\widetilde{V}_j$ associated with the quasi-affine system $\widetilde{X}(\Psi) = \widetilde{X}_0(\Psi)$. The results corresponding to the oversampling rate $q^J \mathbb{N}_0$ can be obtained similarly. So, we consider the quasi-affine system $\widetilde{X}(\Psi)$.

Theorem 3.4. *Let $X(\Psi)$ be the affine tight frame system obtained via the UEP and $\widetilde{X}(\Psi)$ be the quasi-affine frame derived from $X(\Psi)$. Then for all $f \in L^2(K)$, we have*

$$\widetilde{P}_{j+1} f(x) = \widetilde{P}_j f(x) + \sum_{\ell=1}^{L} \sum_{k \in \mathbb{N}_0} \langle f, \widetilde{\psi}^{\ell}_{j,k} \rangle \widetilde{\psi}^{\ell}_{j,k}(x). \tag{3.8}$$

Proof. For $j \geq 0$, we have

$$\widetilde{\varphi}_{j,k} = D_j T_{u(k)} \varphi = \varphi_{j,k} \text{ and } \widetilde{\psi}^{\ell}_{j,k} = D_j T_{u(k)} \psi = \psi^{\ell}_{j,k},$$

which yield

$$P_j f(x) = \sum_{k \in \mathbb{N}_0} \langle f, \varphi_{j,k} \rangle \varphi_{j,k}(x) = \sum_{k \in \mathbb{N}_0} \langle f, \widetilde{\varphi}_{j,k} \rangle \widetilde{\varphi}_{j,k}(x) = \widetilde{P}_j f(x)$$

and

$$Q_j f(x) = \sum_{\ell=1}^{L} \sum_{k \in \mathbb{N}_0} \langle f,\, \psi_{j,k}^{\ell} \rangle \psi_{j,k}^{\ell}(x)$$

$$= \sum_{\ell=1}^{L} \sum_{k \in \mathbb{N}_0} \langle f,\, \widetilde{\psi}_{j,k}^{\ell} \rangle \widetilde{\psi}_{j,k}^{\ell}(x) = \widetilde{Q}_j f(x).$$

Using (3.4), we have

$$\widetilde{P}_{j+1} f(x) = P_{j+1} f(x) = P_j f(x) + \sum_{\ell=1}^{L} \sum_{k \in \mathbb{N}_0} \langle f,\, \psi_{j,k}^{\ell} \rangle \psi_{j,k}^{\ell}(x)$$

$$= \widetilde{P}_j f(x) + \sum_{\ell=1}^{L} \sum_{k \in \mathbb{N}_0} \langle f,\, \widetilde{\psi}_{j,k}^{\ell} \rangle \widetilde{\psi}_{j,k}^{\ell}(x).$$

Thus, (3.8) holds for $j \geq 0$. Let us now prove the result for the case $j < 0$. We also denote φ by ψ^0.

Using equations (3.1) and (3.3), we obtain

$$\psi^{\ell}(x) = \sqrt{q} \sum_{k \in \mathbb{N}_0} h_k^{\ell} \varphi(\mathfrak{p}^{-1} x - u(k)). \tag{3.9}$$

Equation (3.9) leads to

$$\widetilde{\psi}_{j,k}^{\ell}(x) = q^j T_{u(k)} \psi^{\ell}(\mathfrak{p}^{-j} x)$$

$$= q^{j+1} T_{u(k)} \left\{ \sum_{r \in \mathbb{N}_0} h_r^{\ell} \psi^0(\mathfrak{p}^{-j-1} x - u(r)) \right\}$$

$$= \sum_{r \in \mathbb{N}_0} h_r^{\ell} q^{j+1} \psi^0(\mathfrak{p}^{-j-1}(x - u(k) - \mathfrak{p}^{j+1} u(r)))$$

$$= \sum_{r \in \mathfrak{p}^{j+1} \mathbb{N}_0} h_{\mathfrak{p}^{j+1} r}^{\ell} q^{j+1} \psi^0(\mathfrak{p}^{-j-1}(x - u(k) - u(r))).$$

Let us define the dilated sequence h_j^ℓ by

$$h_{j,k}^{\ell} = \begin{cases} h^{\ell}_{\mathfrak{p}^{j+1}k}, & k \in \mathfrak{p}^{j+1}\mathbb{N}_0; \\ 0, & k \notin \mathfrak{p}^{j+1}\mathbb{N}_0. \end{cases}$$

With the dilated sequence defined above, we have

$$\widetilde{\psi}_{j,k}^{\ell}(x) = \sum_{r\in\mathbb{N}_0} h_{r,j}^{\ell}\widetilde{\psi}_{j+1,k+r}^{0}(x).$$

Hence, the right side of the identity (3.8) can be written as

$$\sum_{\ell=0}^{L}\sum_{k\in\mathbb{N}_0}\langle f, \widetilde{\psi}_{j,k}^{\ell}\rangle\widetilde{\psi}_{j,k}^{\ell}(x)$$

$$= \sum_{\ell=0}^{L}\sum_{k\in\mathbb{N}_0}\left\{\sum_{r\in\mathbb{N}_0}\overline{h_{r,j}^{\ell}}\langle f, \widetilde{\psi}_{j+1,k+r}^{0}\rangle\right\}\left\{\sum_{s\in\mathbb{N}_0}h_{s,j}^{\ell}\widetilde{\psi}_{j+1,k+s}^{0}(x)\right\}$$

$$= \sum_{r\in\mathbb{N}_0}\sum_{s\in\mathbb{N}_0}\left\{\sum_{\ell=0}^{L}\sum_{k\in\mathbb{N}_0}\overline{h_{k,j}^{\ell}}h_{k+s-r,j}^{\ell}\right\}\langle f, \widetilde{\psi}_{j+1,r}^{0}\rangle\widetilde{\psi}_{j+1,s}^{0}(x).$$

We first check

$$\sum_{\ell=0}^{L}\sum_{k\in\mathbb{N}_0}\overline{h_{k,j}^{\ell}}h_{k+s-r,j}^{\ell} = \delta_{0,r-s}.$$

When $k - s \in \mathfrak{p}^{j+1}\mathbb{N}_0$, then there exists $\nu \in \mathbb{N}_0$ such that $k - s = \mathfrak{p}^{j+1}\nu$ and we obtain

$$\sum_{\ell=0}^{L}\sum_{k\in\mathbb{N}_0}\overline{h_{k,j}^{\ell}}h_{k+s-r,j}^{\ell} = \sum_{\ell=0}^{L}\sum_{k\in\mathbb{N}_0}\overline{h_{k,j}^{\ell}}h^{\ell}_{k-\mathfrak{p}^{j+1}\nu}$$

$$= \sum_{\ell=0}^{L}\sum_{k\in\mathfrak{p}^{j+1}\mathbb{N}_0}\overline{h_{k,j}^{\ell}}h^{\ell}_{k-\mathfrak{p}^{j+1}\nu}$$

$$= \sum_{\ell=0}^{L} \sum_{k \in \mathbb{N}_0} \overline{h_k^\ell} h_{k-\nu}^\ell$$

$$= \delta_{0,\nu}.$$

The sum is nonzero if and only if $\nu = 0$, which is exactly $r = s$. If $k - s \in \mathfrak{p}^{j+1}\mathbb{N}_0$, then there exist $\nu_1, \nu_2 \in \mathbb{N}_0$ with $\nu_2 \notin \mathbb{N}_0$ such that $k - s = \mathfrak{p}^{j+1}\nu_1 + \nu_2$ and we get

$$\sum_{\ell=0}^{L} \sum_{k \in \mathbb{N}_0} \overline{h_{k,j}^\ell} h_{k+s-r,j}^\ell = \sum_{\ell=0}^{L} \sum_{k \in \mathbb{N}_0} \overline{h_{k,j}^\ell} h_{k-\mathfrak{p}^{j+1}\nu_1-\nu_2}^\ell$$

$$= \sum_{\ell=0}^{L} \sum_{k \in \mathfrak{p}^{j+1}\mathbb{N}_0} \overline{h_{k,j}^\ell} h_{k-\mathfrak{p}^{j+1}\nu_1-\nu_2}^\ell.$$

When $k \in \mathfrak{p}^{j+1}\mathbb{N}_0$, then $k - \mathfrak{p}^{j+1}\nu_1 - \nu_2 \notin \mathfrak{p}^{j+1}\mathbb{N}_0$ and $h_{k-\mathfrak{p}^{j+1}\nu_1-\nu_2}^\ell = 0$ with last identity equalling 0. Hence, for the filters $h_j^1, h_j^2, \ldots, h_j^L$, we have

$$\sum_{\ell=0}^{L} \sum_{k \in \mathbb{N}_0} \overline{h_k^\ell} h_{k-\nu}^\ell = \delta_{0,\nu}, \quad \nu \in \mathbb{N}_0. \tag{3.10}$$

Hence, we get

$$\sum_{\ell=0}^{L} \sum_{k \in \mathbb{N}_0} \langle f, \tilde{\psi}_{j,k}^\ell \rangle \tilde{\psi}_{j,k}^\ell(x) = \sum_{k \in \mathbb{N}_0} \langle f, \tilde{\psi}_{j+1,k}^\ell \rangle \tilde{\psi}_{j+1,k}^\ell(x)$$

$$= \tilde{P}_{j+1} f(x).$$

This proves the identity for $j < 0$. Hence, the result holds for $j \in \mathbb{N}_0$.

Theorem 3.5. *Let* $X(\Psi)$ *be the affine tight frame system obtained via the UEP and* $\widetilde{X}(\Psi)$ *be the quasi-affine frame derived from* $X(\Psi)$. *Then for all* $f \in L^2(K)$, *we have*

$$\widetilde{P}_j f(x) = \widetilde{Q}_j f(x).$$

Proof. We first prove the result for the case $j \geq 0$. Since

$$\widetilde{\varphi}_{j,k} = D_j T_{u(k)} \varphi = \varphi_{j,k},$$

$$P_j f(x) = \sum_{k \in \mathbb{N}_0} \langle f, \varphi_{j,k} \rangle \varphi_{j,k}(x) = \sum_{k \in \mathbb{N}_0} \langle f, \widetilde{\varphi}_{j,k} \rangle \widetilde{\varphi}_{j,k}(x) = \widetilde{P}_j f(x).$$

We now show that $Q_j f(x) = \widetilde{Q}_j f(x)$, for $j \geq 0$. As $X(\Psi)$ is a tight frame, $\widetilde{X}(\Psi)$ is also a tight frame. Further, for $j \geq 0$, we have $\widetilde{\psi}_{j,k} = D_j T_{u(k)} \psi = \psi_{j,k}$.

Hence, we obtain

$$\sum_{\ell=1}^{L} \sum_{j<0} \sum_{k \in \mathbb{N}_0} \langle f, \psi^{\ell}_{j,k} \rangle \psi^{\ell}_{j,k}(x)$$

$$= f(x) - \sum_{\ell=1}^{L} \sum_{j \geq 0} \sum_{k \in \mathbb{N}_0} \langle f, \psi^{\ell}_{j,k} \rangle \psi^{\ell}_{j,k}(x)$$

$$= \sum_{\ell=1}^{L} \sum_{j<0} \sum_{k \in \mathbb{N}_0} \langle f, \widetilde{\psi}^{\ell}_{j,k} \rangle \widetilde{\psi}^{\ell}_{j,k}(x).$$

Therefore, for $j \geq 0$, we have

$$\widetilde{Q}_j f(x)$$

$$= \sum_{\ell=1}^{L} \sum_{u<0} \sum_{k \in \mathbb{N}_0} \langle f, \widetilde{\psi}^{\ell}_{u,k} \rangle \widetilde{\psi}^{\ell}_{u,k}(x) + \sum_{\ell=1}^{L} \sum_{u=0}^{j} \sum_{k \in \mathbb{N}_0} \langle f, \psi^{\ell}_{u,k} \rangle \psi^{\ell}_{u,k}(x)$$

$$= \sum_{\ell=1}^{L} \sum_{u<0} \sum_{k\in\mathbb{N}_0} \langle f, \psi_{u,k}^{\ell} \rangle \psi_{u,k}^{\ell}(x) + \sum_{\ell=1}^{L} \sum_{u=0}^{j} \sum_{k\in\mathbb{N}_0} \langle f, \psi_{u,k}^{\ell} \rangle \psi_{u,k}^{\ell}(x)$$

$$= Q_j f(x).$$

Since $P_j f(x) = Q_j f(x)$, we have $\widetilde{P}_j f(x) = \widetilde{Q}_j f(x)$. This proves the result for the case $j \geq 0$.

We now show that the result also holds for the case $j < 0$. Using Theorem 3.4 inductively for $j < 0$, we have for any $f \in L^2(K)$,

$$\widetilde{P}_j f(x) = \widetilde{P}_v f(x) + \sum_{\ell=1}^{L} \sum_{u=v}^{j} \sum_{k\in\mathbb{N}_0} \langle f, \widetilde{\psi}_{u,k}^{\ell} \rangle \widetilde{\psi}_{u,k}^{\ell}(x). \tag{3.11}$$

Thus, the proof of the result reduces to the proof of $\widetilde{P}_v f(x) \to 0$ as $v \to -\infty$.

Since h_0 is finitely supported, the integer shifts of $\widetilde{\varphi}_v$ provide a Bessel sequence. Since

$$\widetilde{P}_v f(x) = \sum_{k\in\mathbb{N}_0} \langle f, \widetilde{\varphi}_{v,k}^{\ell} \rangle \widetilde{\varphi}_{v,k}^{\ell}(x),$$

the norm of $\widetilde{P}_v f(x)$ satisfies

$$\| \widetilde{P}_v \|_{L^2(K)}^2 \leq C \sum_{k\in\mathbb{N}_0} | \langle f, \widetilde{\varphi}_{v,k}^{\ell} \rangle |^2, \tag{3.12}$$

where the constant C is independent of v. We need to check the value of $\| \widetilde{P}_v \|_{L^2(K)}$ when f is supported on $[-M, M]$ for some $M > 0$. By the Cauchy-Schwartz inequality, we have, for $v < 0$ and $|v|$ sufficiently large,

$$\| \widetilde{P}_v \|_{L^2(K)}^2 \leq C \| f \|_{L^2(K)}^2 \int_{E_v} | \varphi(x) |^2 dx, \tag{3.13}$$

where

$$E_v = \bigcup_{k \in \mathbb{N}_0} (k + \mathfrak{p}^{-v}[-M, M]).$$

As $\widetilde{P}_v f(x) \to 0$ when $v \to -\infty$ in (3.13), the identity (3.11) becomes

$$\widetilde{P}_j f(x) = \widetilde{P}_v f(x) + \sum_{\ell=1}^{L} \sum_{u<j} \sum_{k \in \mathbb{N}_0} \langle f, \widetilde{\Psi}_{u,k}^{\ell} \rangle \widetilde{\Psi}_{u,k}^{\ell}(x) = \widetilde{Q}_j f(x).$$

This completes the proof of the theorem.

4. Conclusion

The Fourier transform due to its deep significance has subsequently been recognized by mathematicians and physicists. Many applications, including the analysis of stationary signals and real-time signal processing, make an effective use of the Fourier transform in time and frequency domains. In this paper, we extended the study based on the extension principles and provided an explicit construction of a class of tight affine frames as well as quasi-affine wavelet frames on local fields of positive characteristic. These results provided a new way of construction of wavelet frames in terms of Fourier transform. They also provided a way for obtaining a new characterization of affine and quasi-affine wavelet frames in terms of low pass and high pass filters.

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