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Wigner–Ville Distribution Associated with Clifford Geometric Algebra $\mathcal{C}l_{n,0}, n = 3(\text{mod } 4)$ Based on Clifford–Fourier Transform

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Abstract: In this study, the Wigner–Ville distribution is associated with the one sided Clifford–Fourier transform over $\mathbb{R}^n$, $n = 3(\text{mod } 4)$. Accordingly, several fundamental properties of the WVD-CFT have been established, including non-linearity, the shift property, dilation, the vector differential, the vector derivative, and the powers of $\tau \in \mathbb{R}^n$. Moreover, powerful results on the WVD-CFT have been derived such as Parseval's theorem, convolution theorem, Moyal's formula, and reconstruction formula. Eventually, we deduce a directional uncertainty principle associated with WVD-CFT. These types of results, as well as methodologies for solving them, have applications in a wide range of fields where symmetry is crucial.

Keywords: Fourier transform; Clifford–Fourier transform; Wigner–Ville distribution; Moyal's formula; uncertainty principle

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1. Introduction

The Fourier transform is one of the most crucial fields in pure and applied mathematics. Recently, the Fourier transform has been widely studied in integral transforms in the real, complex or quaternion setting [1–3]. Brackx et al. [4] extended the Fourier transform to Clifford analysis $\mathcal{C}l_{0,n}$, which is called the Clifford–Fourier transform (CFT). Some characterizations of the CFT have been discussed [5], and its application in the vector fields and vector-valued filters have investigated by Ebling and Scheuremann [6]. Various types of CFT's were intensively explored by many researchers. One of the most studied and investigated versions of CFT is $\mathcal{C}l_{3,0}$ [6]. The geometric algebra of three-dimensional Euclidean space $\mathbb{R}^3$ has been extended to n-dimensional Euclidean space Clifford algebras $\mathcal{C}l_{n,0}$ [7–9], where some fundamental properties such as convolution, correlation, and the uncertainty principle were obtained. Furthermore, some other properties of CFT $\mathcal{C}l_{n,0}$ where $n = 3(\text{mod } 4)$ have also been proved, which involve linearity, scaling, shifting in space and frequency domain, the vector derivative, the vector differential, and the Parseval theorem. The directional uncertainty principle for $\mathcal{C}l_{n,0}$ has also been verified [5]. Some authors presented the CFT differently, as in [10]. Hitzer in [11] proposed a new type of CFT, which can be regarded as the general form of two-sided quaternion Fourier transform (QFT) [12,13].

The Wigner–Ville distribution introduced by J.Ville can be described as one of the most effective methods in detecting linear frequency-modulated (LFM) signals and parameter estimation. The WVD plays a vital role in the analysis of non-stationary signals [14]. Hahn and Snopek developed Fourier- Wigner distributions of 2D quaternion signals [15], and then Bahri thoroughly discussed the 2D WVD associated with QFT [16]. Since then, tremendous work has been done on WVD [17–20]. The idea of associating the WVD with the Clifford algebra of n-dimensional Euclidean space $\mathbb{R}^n$ has not been explored yet. The main purpose of this paper is to investigate the CFT and WVD and to derive the fundamental properties of the WVD associated with the CFT, which include linearity, scaling, shift property, vector differential and derivative, parseval theorem, convolution theorem, correlation theorem, Moyal's formula, and the directional uncertainty principle.

The findings of our work can be best utilized in symmetry. The content of the current paper is organized as follows. Section 2 displays basic notions and results on Clifford algebra, which are needed throughout this study. In Section 3, the results regarding the Clifford–Fourier transform in $\mathbb{R}^3$ are obtained and extended to Clifford–Fourier transform in $\mathbb{R}^n$, $n = 3(\text{mod } 4)$. Section 4 deals with our main findings in detail, that is, the Wigner–Ville distribution associated with Clifford–Fourier transform.

2. Preliminaries

Clifford Geometric Algebra $\mathcal{C}_3$ of $\mathbb{R}^3$

The geometric algebra over $\mathbb{R}^3$ denoted by $\mathcal{C}_3$ consists of a graded $8 = 2^3$ dimensional basis given by

$$\{1, e_1, e_2, e_3, e_{12}, e_{23}, e_{31}, e_{123}\}$$

where $\{e_1, e_2, e_3\}$ is the orthonormal basis of the real 3D Euclidean vector space $\mathbb{R}^3$.

Here, 1 is the real scalar identity element of grade 0; e_1, e_2, e_3 are the basis vectors of $\mathbb{R}^3$ having grade 1; $e_{12} = e_1e_2, e_{23} = e_2e_3$, and $e_{31} = e_3e_1$ are the grade 2 basis bi-vectors that are frequently used; and $e_{123} = e_1e_2e_3 = i_3$ is the trivector or volume element or unit-oriented pseudoscalar having grade 3—the highest grade blade element in $\mathcal{C}_3$, which commutes with all the other elements of $\mathcal{C}_3$ and $i_3^2 = -1$.

The basis vectors obey the following restrictions:

$$\begin{aligned} e_me_n &= -e_ne_m \quad for \quad m \neq n, \quad m, n = 1, 2, 3 \\ e_m^2 &= 1 \qquad for \quad m = 1, 2, 3. \end{aligned}$$

Therefore, inner products obey the following condition:

$$e_m.e_n = \frac{1}{2}(e_me_n + e_ne_m) = \delta_{mn}, \quad m, n = 1, 2, 3.$$

Thus, the inner product of grade 1 vectors x and y is given as

$$\begin{aligned} x.y \quad &= \frac{1}{2}(xy + yx) \\ &= (x_1e_1 + x_2e_2 + x_3e_3).(y_1e_1 + y_2e_2 + y_3e_3) \\ &= x_1y_1 + x_2y_2 + x_3y_3. \end{aligned}$$

Likewise, the outer product of two arbitrary grade 1 vectors x and y is as

$$\begin{aligned} x \wedge y \quad &= \frac{1}{2}(xy - yx) \\ &= (x_1y_2 - x_2y_1)e_{12} + (x_3y_1 - x_1y_3)e_{31} + (x_2x_3 - x_3x_2)e_{23}. \end{aligned}$$

Hence, the Clifford geometric product of two arbitrary grade 1 vectors is written as

$$xy = x.y + x \wedge y, \tag{1}$$

where xy is the scaler quantity and $x \wedge y$ is the vector quantity. Therefore, Equation (1) clearly represents that the Clifford geometric product is the addition of the scaler and vector quantities.

Generally speaking, the elements of a geometric algebra are called multi-vectors. In $\mathcal{C}_3$, every multi-vector M can be expressed as

$$\begin{aligned} P &= \sum_A \alpha_A e_A \quad (2)\\ &= \alpha_0 + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 \\ &+ \alpha_{12} e_{12} + \alpha_{23} e_{23} + \alpha_{31} e_{31} + \alpha_{123} e_{123}. \quad (3) \end{aligned}$$

Equation (3) represents that every multi-vector can be expressed as linear combination of k- grade elements, $k = 0, 1, 2, 3$, where $A \in \{0, 1, 2, 3, 12, 23, 31, 123\}$ and $\alpha \in \mathbb{R}$. The above Equation (3) can be written as

$$P = \langle P \rangle + \langle P \rangle_1 + \langle P \rangle_2 + \langle P \rangle_3 \quad (4)$$

where $\langle P \rangle_k$ is called the grade selector for the k- vector part of M, specifically, $\langle P \rangle = \langle P \rangle_0$. The reverse of P is defined by the anti-automorphism

$$\widetilde{P} = \langle P \rangle + \langle P \rangle_1 - \langle P \rangle_2 - \langle P \rangle_3 \quad (5)$$

which satisfies $\widetilde{(PQ)} = \widetilde{Q}\widetilde{P}$ for every $P, Q \in \mathcal{C}_3$. The square of the norm is defined by

$$\|P\|^2 = \langle P\widetilde{Q} \rangle = \sum_A \alpha_A^2 \quad (6)$$

where

$$\langle P\widetilde{Q} \rangle = P * \widetilde{Q} = \sum_A \alpha_A \beta_A$$

represents real valued scalar product for any multi-vectors $P, Q \in \mathcal{C}_3$, where P and Q are given by

$$P = \sum_A \alpha_A e_A \quad (7)$$

$$Q = \sum_A \beta_A e_A. \quad (8)$$

Note that

$$\langle P \quad Q \rangle = \langle Q \quad P \rangle = \langle \widetilde{P} \quad \widetilde{Q} \rangle = \langle \widetilde{Q} \quad \widetilde{P} \rangle$$

and

$$x^2 \|P\|^2 = \|x\|^2 \|P\|^2 = \|xP\|^2, \qquad x \in \mathbb{R}^3. \quad (9)$$

It has been already proved that the norm satisfies the inequality:

$$\langle P\widetilde{Q} \rangle \leq \|P\| \|Q\| \qquad \forall \quad P, Q \in \mathcal{C}_3. \quad (10)$$

Owing to (10), the Cauchy–Schwarz inequality for multi-vectors can be given as:

$$|\langle P\widetilde{Q} \rangle|^2 \leq \|P\|^2 \|Q\|^2 \qquad \forall \quad P, Q \in \mathcal{C}_3. \quad (11)$$

3. Clifford–Fourier Transform

In this section, we discuss briefly the concept of Fourier transform in $\mathbb{R}$ and extend it to a Clifford's geometric algebra $\mathcal{C}_3$ of dimension 3. Other generalizations can be found in [21–23].

3.1. Fourier Transform in $\mathbb{R}$

The definition of Fourier transform has been given by Popoulis [24] as

Definition 1. *The Fourier transform of an integrable function $f \in L^2(R)$ is the function $\mathcal{F}\{f\}$: $\mathbb{R} \to \mathbb{C}$ defined by*

$$\mathcal{F}\{f\}(\omega) = \int_{\mathbb{R}} f(t)e^{-i\omega t}dt \tag{12}$$

where $i^2 = -1$ and $e^{-i\omega t} = cos\omega t - isin\omega t$.

The general form of the function $\mathcal{F}\{f\}(\omega)$ is given by

$$\mathcal{F}\{f\}(\omega) = A(\omega) + iB(\omega) = C(\omega)e^{i\phi(\omega)} \tag{13}$$

where $C(\omega)$ is called the Fourier spectrum of $f(t)$ and $\phi(\omega)$ is its phase angle.

Definition 2 (Inverse Fourier transform)**.** *The inverse Fourier transform of $f \in L^2(\mathbb{R})$ is defined by*

$$f(t) = \mathcal{F}^{-1}[\mathcal{F}\{f\}](\omega) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathcal{F}\{f\}(\omega)e^{i\omega t}d\omega. \tag{14}$$

The basic properties of the Fourier transform are given below in Table 1.

Now, we will discuss the Clifford geometric algebra Fourier transform in $\mathcal{C}_3$.

Consider a multi-vector valued function $f : \mathbb{R}^3 \to \mathcal{C}_3$, i.e.,

$$\begin{aligned} f(t) &= \sum_A f_A(t)e_A && (15) \\ &= f_0(t) + f_1(t)e_1 + f_2(t)e_2 + f_3e_3 \\ &\quad + f_{12}(t)e_{12} + f_{23}(t)e_{23} + f_{31}e_{31} + f_{123}e_{123}, && (16) \end{aligned}$$

where f_A are real valued functions and t is a vector variable. The above Equation (16) can be expressed as a set of four complex functions

$$\begin{aligned} f(t) &= [f_0(t) + f_{123}(t)i_3] + [f_1(t) + f_{23}(t)i_3]e_1 \\ &\quad + [f_2(t) + f_{31}(t)i_3]e_2 + [f_3(t) + f_{12}(t)i_3]e_3. \end{aligned} \tag{17}$$

This is the motivation behind the extension of Fourier transform to Clifford–Fourier transform (CFT), the definition of which is given below.

Definition 3. *The Clifford–Fourier transform of $f(t) \in L^2(\mathbb{R}^3, \mathcal{C}_3)$ is given by*

$$\mathcal{F}\{f\}(\omega) = \int_{\mathbb{R}^3} f(t)e^{-i_3\omega.t}d^3t, \tag{18}$$

where $t = t_1e_1 + t_2e_2 + t_3e_3$; $\omega = \omega_1e_1 + \omega_2e_2 + \omega_3e_3$; and e_1, e_2, e_3 are the basis vectors of $\mathbb{R}^3$ and

$$d^3t = \frac{dt_1 \wedge dt_2 \wedge dt_3}{i_3} \tag{19}$$

is scalar valued, where $dt_k = d\boldsymbol{t}_k e_k, k = 1, 2, 3$ no summation.

Since i_3 commutes with element of $\mathcal{C}_3$, the Clifford–Fourier Kernel $e^{-i_3\omega.t}$ also commutes with every element of $\mathcal{C}_3$

Theorem 1 (Inverse Clifford–Fourier transform)**.** *Let $f \in L^2(\mathbb{R}^3, \mathcal{C}_3)$ with $\int_{\mathbb{R}^3} \|f\|^2 d^3t < \infty$; then, the inverse of Clifford–Fourier transform is calculated by*

$$f(t) = \mathcal{F}^{-1}[\mathcal{F}\{f\}](t) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \mathcal{F}f(\omega)e^{i_3\omega.t}d^3\omega. \tag{20}$$

Equation (20) is called the Clifford–Fourier integral theorem. It just shows how to get back to the given function f from the Fourier transform.

The proof is already formulated in [5].

The important properties and some theorems related to the CFT are summarized in Table 1, which have been already proved in [5].

Table 1. Properties of 3-dimensional Clifford–Fourier transform $f, g \in L^2(\mathbb{R}^3, \mathcal{C}_3), \alpha, \beta \in \mathcal{C}_3, s \neq 0, \omega_0 \in \mathbb{R}^3$.

Property	Function	CFT
Linearity	$\alpha f(t) + \beta g(t)$	$\alpha\mathcal{F}\{f\}(\omega) + \beta\mathcal{F}\{f\}(\omega)$
Delay property	$f_d(t) = f(t-a)$	$e^{-i_3\omega.a}\mathcal{F}\{f\}(\omega)$
Scaling property	$f_s(t) = f(st)$	$\frac{1}{s^3}\mathcal{F}\{f\}(\frac{\omega}{s})$
Shift property	$f_0(t) = f(t)e^{i_3\omega_0.t}$	$\mathcal{F}\{f\}(\omega - \omega_0)$
Vector differential	$a.\nabla f$	$i_3 a.\omega\mathcal{F}\{f\}(\omega)$
Vector derivative	∇f	$i_3\omega\{f\}(\omega)$
Convolution	$f * g$	$\mathcal{F}\{f\}(\omega)\mathcal{F}\{g\}(\omega)$
Parseval theorem	$\int_{\mathbb{R}^3} \|f(t)\|^2 d^3t$	$\frac{1}{(2\pi)^3}\|\mathcal{F}\{f\}(\omega)\|^2 d^3\omega$

3.2. Generalization towards One Sided n-Dimensional Clifford–Fourier Transform

This section recalls the definition of n-dimensional Clifford–Fourier transform $\mathcal{C}_n = Cl(n,0), n = 3(\text{mod } 4)$ with a graded 2^n dimensional basis. Then, we present some important properties of $\mathcal{C}_n = Cl(n,0)$ [5]. For more details on the Clifford algebra of the n dimension, see reference [5].

Definition 4. *The Clifford–Fourier transform of $f(t) \in L^2(\mathbb{R}^n, \mathcal{C}_n)$ is given by:*

$$\mathcal{F}\{f\}(\omega) = \int_{\mathbb{R}^n} f(t)e^{-i_n\omega.t}d^n t, \tag{21}$$

where $t = t_1e_1 + t_2e_2 + t_3e_3 + \ldots + t_ne_n$; $\omega = \omega_1e_1 + \omega_2e_2 + \omega_3e_3 + \ldots + \omega_ne_n$; and $e_1, e_2, e_3 \ldots, e_n$ are the basis vectors of $\mathbb{R}^n$
and

$$d^n t = \frac{dt_1 \wedge dt_2 \wedge dt_3 \ldots \wedge dt_n}{i_n}, \tag{22}$$

is scalar valued, where $dt_k = d\mathbf{t}_k e_k, k = 1,2,3,\ldots,n$ no summation.

Since i_n commutes with the element of $\mathcal{C}_n$, for $n = 3(\text{mod } 4)$, the Clifford–Fourier Kernel $e^{-i_n\omega.t}$ also commutes with every element of $\mathcal{C}_n$. However, this is not true in the case of $n = 2(\text{mod } 4)$.

Theorem 2 (Inverse Clifford–Fourier transform)**.** *Let $f \in L^2(\mathbb{R}^n, \mathcal{C}_n)$ with $\int_{\mathbb{R}^n} \|f\|^2 d^n t < \infty$; then, the inverse of Clifford–Fourier transform is calculated by*

$$f(t) = \mathcal{F}^{-1}[\mathcal{F}\{f\}](t) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \mathcal{F}\{f\}(\omega)e^{i_n\omega.t}d^n\omega. \tag{23}$$

Proof. The proof has already been derived in [7] for the 3-dimensional case, which can be easily generalized in the case of n-dimensional Clifford–Fourier transform. □

Now, we will present some properties of $\mathcal{C}_n = Cl(n,0)$ in the Table 2, which will satisfy for $n = 3$ (mod 4) due to the commutative property of i_n for $n = 3$ (mod 4).

Table 2. Properties of n-dimensional Clifford–Fourier transform, $n = 3(\text{mod } 4)$, $f, g \in L^2(\mathbb{R}^n, \mathcal{C}_n), \alpha, \beta \in \mathcal{C}_n, a \neq 0, \omega_0 \in \mathbb{R}^n$.

Property	Function	CFT
Left linearity	$\alpha f(t) + \beta g(t)$	$\alpha\mathcal{F}\{f\}(\omega) + \beta\mathcal{F}\{f\}(\omega)$
Scaling	$f_a(t) = f(at), a \neq 0$	$\frac{1}{\lvert a\rvert^n}\mathcal{F}\{f\}(\frac{\omega}{a})$
Shift in frequency domain	$f_0(t) = f(t)e^{i_n\omega_0.t}$	$\mathcal{F}\{f\}(\omega - \omega_0)$
Shift in space domain	$f_0(t) = f(t-a)$	$\mathcal{F}\{f\}(\omega)e^{-i_n\omega.a}$
Power of $t \in \mathbb{R}^n$ from left	$t^m f(t)$	$\nabla^m_\omega \mathcal{F}\{f\}(\omega) i_n^m$
Power of $t \in \mathbb{R}^n$ from right	$f(t)t^m$	$\int_{\mathbb{R}^n} f(t)\nabla^m_\omega e^{-i_n\omega.t} d^n t. i_n^m$
Parseval theorem	$\int_{\mathbb{R}^n} f_1(t)\widetilde{f_2(t)} d^n t$	$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \mathcal{F}\{f_1\}(\omega)\widetilde{\mathcal{F}\{f_2\}(\omega)} d^n t$
Scalar Parseval theorem	$\int_{\mathbb{R}^n} \lVert f(t)\rVert^2 d^n t$	$\frac{1}{(2\pi)^n}\lVert\mathcal{F}\{f\}(\omega)\rVert^2 d^n\omega$
Convolution	$f * g$	$\mathcal{F}\{f\}(\omega)\mathcal{F}\{g\}(\omega)$
Vector derivative (left)	$\nabla^m f(t)$	$i_n^m \omega^m \mathcal{F}\{f\}(\omega)$
Vector derivative (right)	$f(t)\nabla^m$	$i_n^m \mathcal{F}\{f\}(\omega)\omega^m$

In the coming section, we are now going to discuss our main work, that is, the Wigner–Ville distribution associated with the n-dimensional Clifford–Fourier transform, where $n = 3(\text{mod } 4)$.

4. Wigner–Ville Distribution Associated with Clifford Geometric Algebra $\mathcal{C}l_{n,0}, n = 3(\text{mod } 4)$ Based on Clifford–Fourier Transform, $n = 3(\text{mod } 4)$

We begin by providing the definition of the Wigner–Ville distribution and enlist its properties.

Definition 5. *The Wigner–Ville transform (WVT) of $f, g \in L^2(\mathbb{R})$ is defined by:*

$$W_{f,g}(t,\omega) = \int_{\mathbb{R}} f\left(t + \frac{\tau}{2}\right)\overline{g\left(t - \frac{\tau}{2}\right)} e^{-i\omega\tau} d\tau. \tag{24}$$

The fundamental properties of WVT and their proofs can be found in [21–23].

Definition 6 (WVD-CFT)**.** *The WVD-CFT of two functions $f, g \in (\mathbb{R}^n, \mathcal{C}_n), n = 3(\text{mod } 4)$ for any $\tau \in \mathbb{R}^n$ is defined as:*

$$\mathcal{W}_{f,g}(t,\omega) = \int_{\mathbb{R}^n} f\left(t + \frac{\tau}{2}\right)\overline{g}\left(t - \frac{\tau}{2}\right) e^{-i_n\boldsymbol{\omega}.\boldsymbol{\tau}} d^n\boldsymbol{\tau}. \tag{25}$$

Suppose the auto correlation is defined as:

$$h_t(\tau) = f\left(t + \frac{\tau}{2}\right)\overline{g}\left(t - \frac{\tau}{2}\right). \tag{26}$$

Therefore (6) becomes

$$W_{f,g}(t,\omega) = \int_{\mathbb{R}^n} h_t(\tau) e^{-i_n\boldsymbol{\omega}.\boldsymbol{\tau}} d^n\boldsymbol{\tau} \tag{27}$$

with $t, \omega \in \mathbb{R}^n$.

Note that $d^n\boldsymbol{\tau} = \dfrac{d\boldsymbol{\tau}_1 \wedge d\boldsymbol{\tau}_2 \wedge d\boldsymbol{\tau}_3 \ldots \wedge d\boldsymbol{\tau}_n}{i_n}$ is scalar valued with $d\boldsymbol{x_k} = dx_k\boldsymbol{e_k} \in \mathbb{R}^n$, $\boldsymbol{k} = 1, 2, \ldots, n$, no summation.

Theorem 3. *The WVD-CFT of* $f \in L^2(\mathbb{R}^n, \mathcal{C}_n), n = 3(mod4)$, *with* $\int_{\mathbb{R}^n} \|f\|^2 d^n\boldsymbol{\tau} < \infty$ *is invertible, and its inverse is calculated by*

$$f\Big(t+\frac{\tau}{2}\Big)\overline{f}\Big(t-\frac{\tau}{2}\Big) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} W_f(t,\omega)e^{i_n\boldsymbol{\omega}.\boldsymbol{\tau}}d^n\boldsymbol{\omega} \tag{28}$$

where $h_t(\tau)$ *is defined in (26).*

Proof.

$$\begin{aligned}
&\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} W_f(t,\omega)e^{i_n\boldsymbol{\omega}.\boldsymbol{\tau}}d^n\boldsymbol{\omega}\\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n} f\Big(y+\frac{\tau}{2}\Big)\overline{f}\Big(y-\frac{\tau}{2}\Big)e^{-i_n\boldsymbol{\omega}.y}d^ny\right]e^{i_n\boldsymbol{\omega}.\boldsymbol{\tau}}d^n\boldsymbol{\omega}\\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f\Big(y+\frac{\tau}{2}\Big)\overline{f}\Big(y-\frac{\tau}{2}\Big)e^{i_n(\tau-y)\omega}d^n\omega d^ny\\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f\Big(y+\frac{\tau}{2}\Big)\overline{f}\Big(y-\frac{\tau}{2}\Big)\prod_{m=1}^{n} e^{i_m(\tau_m-y_m)\omega_m}d^n\omega d^ny\\
&= \int_{\mathbb{R}^n} f\Big(y+\frac{\tau}{2}\Big)\overline{f}\Big(y-\frac{\tau}{2}\Big)\prod_{m=1}^{n}\delta(\tau_m-y_m)d^ny\\
&= f\Big(t+\frac{\tau}{2}\Big)\overline{f}\Big(t-\frac{\tau}{2}\Big).
\end{aligned}$$

This completes the proof.

Note that we have used $\frac{1}{2\pi}\int_{\mathbb{R}} e^{i(\tau_m-y_m)\omega_m}d\omega_m = \delta(\tau_m-y_m), 1 \le m \le n$ for the fourth equality. □

Theorem 4 (Non-linearity property)**.** *Let* $f, g \in L^2(\mathbb{R}^n, \mathcal{C}_n)$, *then*

$$W_{f+g}(t,\omega) = W_f(t,\omega) + W_g(t,\omega) + W_{f,g}(t,\omega)W_{g,f}(t,\omega).$$

Proof.

$$\begin{aligned}
W_{f+g}(t,\omega) &= \int_{\mathbb{R}^n}(f+g)\Big(t+\frac{\tau}{2}\Big)\overline{(f+g)}\Big(t-\frac{\tau}{2}\Big)e^{-i_n\boldsymbol{\omega}.\boldsymbol{\tau}}d^n\boldsymbol{\tau}\\
&= \int_{\mathbb{R}^n}\left[f\Big(t+\frac{\tau}{2}\Big)+g\Big(t+\frac{\tau}{2}\Big)\right]\left[\overline{f\Big(t+\frac{\tau}{2}\Big)}+\overline{g\Big(t+\frac{\tau}{2}\Big)}\right]e^{-i_n\boldsymbol{\omega}.\boldsymbol{\tau}}d^n\boldsymbol{\tau}\\
&= \int_{\mathbb{R}^n} f\Big(t+\frac{\tau}{2}\Big)\overline{f}\Big(t-\frac{\tau}{2}\Big)e^{-i_n\boldsymbol{\omega}.\boldsymbol{\tau}}d^n\boldsymbol{\tau} + \int_{\mathbb{R}^n} g\Big(t+\frac{\tau}{2}\Big)\overline{g}\Big(t-\frac{\tau}{2}\Big)e^{-i_n\boldsymbol{\omega}.\boldsymbol{\tau}}d^n\boldsymbol{\tau}\\
&+ \int_{\mathbb{R}^n} f\Big(t+\frac{\tau}{2}\Big)\overline{g}\Big(t-\frac{\tau}{2}\Big)e^{-i_n\boldsymbol{\omega}.\boldsymbol{\tau}}d^n\boldsymbol{\tau} + \int_{\mathbb{R}^n} g\Big(t+\frac{\tau}{2}\Big)\overline{f}\Big(t-\frac{\tau}{2}\Big)e^{-i_n\boldsymbol{\omega}.\boldsymbol{\tau}}d^n\boldsymbol{\tau}\\
&= W_f(t,\tau) + W_g(t,\tau) + W_{f,g}(t,\tau) + W_{g,f}(t,\tau).
\end{aligned}$$

□

Hence, proved.

Theorem 5 (Shift in space domain)**.** *Let* $f \in L^2(\mathbb{R}^n, \mathcal{C}_n)$ *and if* $f_s(t) = f(t-t_0)$, *then*

$$W_{f_s}(t,\omega) = W_f(t-t_0,\omega). \tag{29}$$

Proof.

$$W_{f_s}(t,\omega) = \int_{\mathbb{R}^n} f\Big(t-t_0+\frac{\tau}{2}\Big)\overline{f}\Big(t-t_0-\frac{\tau}{2}\Big)e^{-i_n\boldsymbol{\omega}.\boldsymbol{\tau}}d^n\boldsymbol{\tau}.$$

Put $t - t_0 = \alpha$, we have

$$\begin{aligned} W_{f_s}(t,\omega) &= \int_{\mathbb{R}^n} f\Big(\alpha + \frac{\tau}{2}\Big)\overline{f}\Big(\alpha - \frac{\tau}{2}\Big)e^{-i_n\omega.\tau}d^n\boldsymbol{\tau} \\ &= W_f(\alpha,\omega) \\ &= W_f(t - t_0,\omega). \end{aligned}$$

□

Theorem 6 (Shift in Frequency Domain)**.** *Let* $f \in L^2(\mathbb{R}^n, \mathcal{C}_n)$ *and if* $f_0(t) = f(t)e^{i_n\omega_0 t}$, *then*

$$W_{f_0}(t,\omega) = W_f(t,\omega - \omega_0). \tag{30}$$

Proof.

$$\begin{aligned} W_{f_0}(t,\omega) &= \int_{\mathbb{R}^n} f\Big(t + \frac{\tau}{2}\Big)e^{i_n\omega_0(t+\frac{\tau}{2})}\overline{f}\Big(t - \frac{\tau}{2}\Big)e^{-i_n\omega_0(t-\frac{\tau}{2})}e^{-i_n\omega.\tau}d^n\tau \\ &= \int_{\mathbb{R}^n} f\Big(t + \frac{\tau}{2}\Big)\overline{f}\Big(t - \frac{\tau}{2}\Big)e^{-i_n(\omega-\omega_0)\tau}d^n\tau \\ &= W_f(t,\omega - \omega_0). \end{aligned}$$

□

The WVD-CFT $W_f(t,\omega - \omega_0)$ is centered at the point $\omega = \omega_0$ in the frequency domain.

Theorem 7 (Plancherel theorem for WVD-CFT)**.** *Let* $f, g \in (\mathbb{R}^n, \mathcal{C}_n)$ *with their WVD-CFT's* $W_f(t,\tau)$ *and* $W_g(t,\tau)$, *respectively, then*

$$\int_{\mathbb{R}^n} h_f(t,\tau)\overline{h_g(t,\tau)}d^n\tau = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} W_f(t,\tau)\overline{W_g(t,\tau)}d^n\omega \tag{31}$$

where

$$\begin{aligned} h_f(t,\tau) &= f\Big(t + \frac{\tau}{2}\Big)\overline{f}\Big(t - \frac{\tau}{2}\Big) \\ h_g(t,\tau) &= g\Big(t + \frac{\tau}{2}\Big)\overline{g}\Big(t - \frac{\tau}{2}\Big). \end{aligned}$$

Proof.

$$\begin{aligned} \int_{\mathbb{R}^n} h_f(t,\tau)\overline{h_g(t,\tau)}d^n\tau &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}_n}\Big[\int_{\mathbb{R}_n} W_{f,t}(\tau)e^{i_n\omega.\tau}d^n\omega\Big]\overline{h_{g,t}(\tau)}d^n\tau \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} W_{f,t}(\tau)\Big[\int_{\mathbb{R}^n} h_{g,t}(\tau)e^{-i_n\omega.\tau}d^n\tau\Big]d^n\omega \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} W_f(t,\tau)\overline{W_g(t,\tau)}d^n\omega \end{aligned}$$

which proves (31). □

It is worth mentioning that the Plancherel theorem is a multivector-valued theorem. It is valid for each grade $k, 0 \leq k \leq n$ of the multivectors on both sides of Equation (31). Hence, we conclude the following result.

Corollary 1.

$$\Big\langle \int_{\mathbb{R}^n} h_{f,t}(\tau)\overline{h_{g,t}(\tau)}d^n\tau \Big\rangle_k = \frac{1}{(2\pi)^n}\Big\langle \int_{\mathbb{R}^n} W_{f,t}(\tau)\overline{W_{g,t}(\tau)}d^n\omega \Big\rangle_k. \tag{32}$$

Remark 1. *If* $f = g$, *then we have the following multivector version of the Parseval theorem.*

Theorem 8 (Parseval theorem)**.**

$$\int_{\mathbb{R}^n} h_{f,t}(\tau)\overline{h_{f,t(\tau)}}d^n\tau = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} W_{f,t}(\tau)\overline{W_{f,t}(\tau)}d^n\omega.$$

Since $\|M\|^2 = \langle M\overline{M}\rangle$, the scalar part of the Parseval theorem together with $\|M\|^2 = \langle M\overline{M}\rangle$ gives us the scalar Parseval theorem.

Theorem 9 (Scalar parseval)**.**

$$\int_{\mathbb{R}^n} \|h_{f,t}(\tau)\|^2 d^n\tau = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \|W[h_{f,t}(\tau)]\|^2 d^n\omega \tag{33}$$

Now, we will derive the most important result, that is, convolution for WVD-CFT.

4.1. Convolution for WVD-CFT

In the begininig, we shall define Clifford convolution.

Definition 7. *Let $f,g \in L^2(\mathbb{R}^n, \mathcal{C}_n)$; then, the Clifford convolution is defined by*

$$(f*g)(t) = \int_{\mathbb{R}^n} f(x)g(t-x)dx. \tag{34}$$

Theorem 10 (Convolution for WVD-CFT)**.** *Let $f,g \in L^2(\mathbb{R}^n, \mathcal{C}_n)$; then, the Clifford Convolution for Wigner–Ville associated with CFT is*

$$W_{f*g}(t,\omega) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} W_f(u,\omega)W_g(t-u,\omega)d^n u. \tag{35}$$

Proof. Since

$$W_{f,g}(t,\omega) = \int_{\mathbb{R}^n} f\Big(t+\frac{\tau}{2}\Big)\overline{g}\Big(t-\frac{\tau}{2}\Big)e^{-i_n\boldsymbol{\omega}.\boldsymbol{\tau}}d^n\boldsymbol{\tau}, \tag{36}$$

it follows that

$$\begin{aligned} W_{f*g}(t,\omega) &= \int_{\mathbb{R}^n}(f*g)\Big(t+\frac{\tau}{2}\Big)(\overline{f}*\overline{g})\Big(t-\frac{\tau}{2}\Big)e^{-i_n\omega.\tau}d^n\tau \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)g\Big(t+\frac{\tau}{2}-x\Big)dx \times \int_{\mathbb{R}^n}\overline{f(y)}g\Big(t-\frac{\tau}{2}-y\Big)dy \quad e^{-i_n\omega.\tau}d^n\tau \end{aligned}$$

Put

$$\begin{aligned} x &= u+\frac{v}{2} \\ y &= u-\frac{v}{2} \\ \tau &= v+w. \end{aligned}$$

Therefore,

$$\begin{aligned} W_{f*g}(t,\omega) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f\Big(u+\frac{v}{2}\Big)g\Big(t+\frac{\tau}{2}-\Big(u+\frac{v}{2}\Big)\Big) \\ &\quad \times\overline{f\Big(u-\frac{v}{2}\Big)g\Big((t-\frac{\tau}{2})-(u-\frac{v}{2})\Big)} \quad e^{-i_n\omega.\tau}d^n v d^n w d^n u \\ &= \int_{\mathbb{R}^n}\Big[\int_{\mathbb{R}^n} f\Big(u+\frac{v}{2}\Big)\overline{f\Big(u-\frac{v}{2}\Big)}e^{-i_n\omega.v}d^n v \\ &\quad \times\int_{\mathbb{R}^n} g\Big(t-u+\frac{w}{2}\Big)\overline{g\Big(t-u-\frac{w}{2}\Big)}e^{-i_n\omega.wd^w}\Big]d^n u \\ &= \int_{\mathbb{R}^n} W_f(u,\omega)W_g(t-u,\omega)d^n u \end{aligned}$$

which proves the convolution for WVD-CT. □

The following theorem, that is, the reconstruction formula for the WVD-CFT determines that the Clifford signal can be uniquely determined in terms of the WVD-CFT within a constant factor.

Theorem 11 (Reconstruction formula for WVD-CFT)**.** *The inverse transform of the Clifford signal $f \in L^2(\mathbb{R}^n, \mathcal{C}_n)$ is given by*

$$f(u) = \frac{1}{(2\pi)^2 \overline{g}(0)} \int_{\mathbb{R}^n} W_{f,g}\left(\frac{u}{2}, \omega\right) e^{i_n \omega . u} d^n \omega.$$

provided $\overline{g}(0) \neq 0$.

Proof. Since we know that

$$W_{f,g}(t,\omega) = \int_{\mathbb{R}^n} f\left(t + \frac{\tau}{2}\right) \overline{g}\left(t - \frac{\tau}{2}\right) e^{-i_n \omega . \tau} d^n . \tau.$$

By applying inverse of WVD-CFT (27), we obtain

$$f\left(t + \frac{\tau}{2}\right) \overline{g}\left(t - \frac{\tau}{2}\right) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} W_{f,g}(t,\omega) e^{i_n \omega . \tau} d^n \omega$$

Using the change of variables $\frac{\tau}{2} = t$, we obtain

$$f(2t)\overline{g}(0) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^n} W_{f,g}(t,\omega) e^{2 i_n \omega . t} d^n \omega. \tag{37}$$

Again, using the change of variables $2t = u$, we have

$$f(u) = \frac{1}{(2\pi)^2 \overline{g}(0)} \int_{\mathbb{R}^n} W_{f,g}\left(\frac{u}{2}, \omega\right) e^{i_n \omega . u} d^n \omega.$$

This ends the proof of the Theorem. □

Theorem 12 (Moyal's Formula for WVD-CT)**.** *Let $f, g \in L^2(\mathbb{R}^n, \mathcal{C}_n)$; then, the following equation holds:*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| W_{f,g}(t,\omega) \right|^2 d^n \omega d^n t = (4\pi)^n \int_{\mathbb{R}^n} |f(u)|^2 d^n u \int_{\mathbb{R}^n} |g(v)|^2 d^n v.$$

Proof.

$$W_{f,g}(t,w) = \mathcal{F}_c\{h_t(\tau)\}(w),$$

where $h_t(\tau) = f(t + \frac{\tau}{2})\bar{g}(t - \frac{\tau}{2})$, which implies

$$\begin{aligned}
\int_{\mathbb{R}^n} \left| W_{f,g}(t,w) \right|^2 d^n w &= \|\mathcal{F}_c\{h_t(\tau)\}\|^2_{L^2(\mathbb{R}^n, Cl(n,0))} \\
&= (2\pi)^n \|h_t(\tau\|^2 \\
&= (2\pi)^n \int_{\mathbb{R}^n} f(t + \frac{\tau}{2})\bar{g}(t - \frac{\tau}{2})\overline{f(t + \frac{\tau}{2}).\bar{g}(t - \frac{\tau}{2})} d^n \tau \\
&= (2\pi)^n \int_{\mathbb{R}^n} f(t + \frac{\tau}{2})\bar{g}(t - \frac{\tau}{2})\bar{f}(t + \frac{\tau}{2}).g(t - \frac{\tau}{2}) d^n \tau.
\end{aligned}$$

Integrating above w.r.t $d^n t$, we have

$$\int_{\mathbb{R}^n} \left| W_{f,g}(t,w) \right|^2 d^n w d^n t$$
$$= (2\pi)^n \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(t+\frac{\tau}{2})\bar{g}(t-\frac{\tau}{2})\bar{f}(t+\frac{\tau}{2}).g(t-\frac{\tau}{2})d^n\tau d^n t.$$

On setting $t+\frac{\tau}{2}=u$ and $t-\frac{\tau}{2}=v$, which gives $du=\frac{1}{2}d\tau$ and $dt=dv$

$$\begin{aligned} \int_{\mathbb{R}^n} \left| W_{f,g}(t,w) \right|^2 d^n w d^n t &= (2\pi)^n.2^n \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(u)\bar{g}(v)g(v)\bar{f}(u)d^n v d^n u \\ &= (4\pi)^n \int_{\mathbb{R}^n} f(u)\bar{f}(u)d^n u \int_{\mathbb{R}^n} |g(v)|^2 d^n v \\ &= (4\pi)^n \int_{\mathbb{R}^n} |f(u)|^2 d^n u. \int_{\mathbb{R}^n} |g(v)|^2 d^n v. \end{aligned}$$

This proves the theorem. □

Theorem 13 (Dilation)**.** *Let $f, g \in (\mathbb{R}^n, \mathcal{C}_n)$; then,*

$$W_{D_{cf},D_{cg}} = W_{f,g}\left(\frac{t}{c}, c\omega\right).$$

Proof.

$$W_{D_{cf},D_{cg}} = \frac{1}{c^2}\int_{\mathbb{R}^n} f\left(\frac{t}{c}+\frac{\tau}{2c}\right)\bar{g}\left(\frac{t}{c}-\frac{\tau}{2c}\right)e^{-in\omega.\tau}d^n\tau$$

take $\frac{\tau}{c}=y$, which gives $dy=d\tau$.

Therefore,

$$\begin{aligned} W_{D_{cf},D_{cg}} &= \int_{\mathbb{R}^n} f\left(\frac{t}{c}+\frac{x}{2}\right)\bar{g}\left(\frac{t}{c}-\frac{x}{2}\right)e^{-inc\omega.x}d^n x \\ &= W_{f,g}\left(\frac{t}{c}, c\omega\right) \end{aligned}$$

□

Theorem 14 (Powers of $\boldsymbol{\tau} \in \mathbb{R}^n$ from left)**.**

$$W\{\tau^m h_{f,t}(\tau)\} = \nabla^m_\omega W\{h_{f,t}(\tau)\}i^m_n, \qquad m \in \mathbb{N} \tag{38}$$

where

$$h_t(\tau) = f\left(t+\frac{\tau}{2}\right)\overline{f}\left(t-\frac{\tau}{2}\right). \tag{39}$$

Proof. First we shall prove the theorem for $m=1$

$$\begin{aligned} W\{\tau h_t(\tau)\} &= \int_{\mathbb{R}^n} \tau h_{f,t}(\tau)e^{-i_n\omega.\tau}d^n\tau \\ &= \int_{\mathbb{R}^n} \nabla_\omega h_{f,t}(\tau)i_n e^{-i_n\omega.\tau}d^n\tau \\ &= \nabla_\omega \int_{\mathbb{R}^n} h_{f,t}(\tau)e^{-i_n\omega.\tau}d^n\tau i_n \\ &= \nabla_\omega w\{h_{f,t}(\tau)\}i_n. \end{aligned}$$

Repeating the process $m-1$ times, one obtains

$$W\{\tau^m h_{f,t}(\tau)\} = \nabla^m_\omega W\{h_{f,t}(\tau)\}i^m_n, \qquad m \in \mathbb{N}.$$

□

Theorem 15 (Powers of $\boldsymbol{\tau} \in \mathbb{R}^n$ from the right)**.**

$$W\{h_{f,t}(\tau)\tau^m\} = \int_{\mathbb{R}^n} h_{f,t}(\tau)\nabla_\omega^m e^{-i_n\omega.\tau} d^n\tau \quad i_n^m \qquad m \in \mathbb{N}. \tag{40}$$

Proof. We omit its proof as it follows directly from Theorem 14. □

Now, we will derive the final formulas for the WVD-CFT Of $h_{f,t}(\tau)$ by using the above Equation (40) and the dimension dependent commutation properties of i_n.

Theorem 16.

$$W\{(a.\tau)^m h_{f,t}(\tau)\} = (a.\nabla_\omega)^m W\{h_{f,t}(\tau)\} i_n^m \tag{41}$$

Proof. First, we shall prove the theorem for $m = 1$

$$\begin{aligned} W\{(a.\tau)h_{f,t}(\tau)\} \qquad &= \int_{\mathbb{R}^n} a.\tau h_{f,t}(\tau)e^{-i_n\omega.\tau}d^n\tau \\ &= \int_{\mathbb{R}^n} h_{f,t}(\tau)a.\tau e^{-i_n\omega.\tau}d^n\tau \\ &= \int_{\mathbb{R}^n} h_{f,t}(\tau)a.\nabla_\omega i_n e^{-i_n\omega.\tau}d^n\tau \\ &= a.\nabla_\omega \int_{\mathbb{R}^n} h_{f,t}(\tau)e^{-i_n\omega.\tau}d^n\tau i_n \\ &= a.\nabla_\omega W\{h_{f,t}(\tau)\}i_n. \end{aligned}$$

By repeating the process $m - 1$ times

$$W\{(a.\tau)^m h_{f,t}(\tau)\} = (a.\nabla_\omega)^m W\{h_{f,t}(\tau)\} i_n^m.$$

□

Corollary 2. *On setting $b.\tau h_{f,t}(\tau), b \in \mathbb{R}^n$ for $h_{f,t}(\tau)$, we obtain the following result.*

$$W\{(a.\tau)b.\tau h_{f,t}(\tau)\} = a.\nabla_\omega b.\nabla_\omega W\{h_{f,t}(\tau)\}. \tag{42}$$

Theorem 17 (Vector differential)**.** *The Clifford–Fourier transform of the mth power vector differential of the auto-co-relation function $h_{f,t}(\tau)$ is*

$$W\{(a.\nabla)^m h_{f,t}(\tau)\} = (a.\omega)^m W\{h_{f,t}(\tau)\} i_n^m. \tag{43}$$

Proof. We shall first prove (43) for $m = 1$

$$\begin{aligned} a.\nabla h_{f,t}(\tau) \qquad &= a.\nabla.\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} W\{h_{f,t}(\tau)\}e^{i_n\omega.\tau}d^\tau \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} W\{h_{f,t}(\tau)\}a.\nabla e^{i_n\omega.\tau}d^\tau \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} W\{h_{f,t}(\tau)\}a.\omega i_n e^{i_n\omega.\tau}d^\tau \\ &= W^{-1}\{a.\omega W\Big(h_{f,t}(\tau)\Big)i_n\}. \end{aligned}$$

Therefore,

$$W\{(a.\nabla)h_{f,t}(\tau)\} = (a.\omega)W\{h_{f,t}(\tau)\}i_n. \tag{44}$$

Repeating the above process, we obtain (43). This completes the proof. □

Theorem 18 (Left vector derivative).

$$W\{\nabla^m h_{f,t}(\tau)\} = \omega^m W\{h_{f,t}(\tau)\} i_n^m \tag{45}$$

Proof. The proof is omitted as it follows directly from Theorem 17. □

Theorem 19 (∇^m from right).

$$h_{f,t}(\tau)\nabla^m = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} W\{h_{f,t}(\tau)\} i_n^m e^{i_n\omega.\tau}\omega^m d^n\omega. \tag{46}$$

Proof. Setting $m = 1$

$$\begin{aligned}
h_{f,t}(\tau)\nabla \quad &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} W\{h_{f,t}(\tau)\} e^{i_n\omega.\tau} d^n\omega\nabla \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} W\{h_{f,t}(\tau)\} e^{i_n\omega.\tau}\nabla d^n\omega \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} W\{h_{f,t}(\tau)\} e^{i_n\omega.\tau} i_n\omega d^n\omega \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} W\{h_{f,t}(\tau)\} i_n e^{i_n\omega.\tau}\omega d^n\omega
\end{aligned}$$

This proves the case for $m = 1$. Following the same procedure, we obtain (46). □

4.2. Uncertainty Principle

The uncertainty principle plays a central role in understanding the concepts of quantum physics and is also vital for information processing. Quantum physics states that the conjugate properties such as particle momentum and position cannot be simultaneously measured accurately. Fourier analysis states that a function and its Fourier transform cannot be simultaneously sharply localized.

In order to prove the directional uncertainty principle for WVD-CFT, we first state a lemma and a proposition about the directional uncertainty principle for Clifford–Fourier transform, which has been already proved in [5].

Lemma 1 (Directional uncertainty principle for CFT). *Let $f \in (\mathbb{R}^n, \mathcal{C}_n)$, having Clifford–Fourier transform $\mathcal{F}\{f\}$ with $\int_{\mathbb{R}^n} \|f\|^2 d^n x = F < \infty$; then, for any arbitrary constant vectors a, b:*

$$\int_{\mathbb{R}^n}(a.x)^2\|f(x)\|^2 d^n x.\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}(b.\omega)^2\|\mathcal{F}\{f\}(\omega)\|^2 d^n\omega \geq (a.b)\frac{1}{4}F^2$$

Proposition 1 (Integration by parts).

$$\int_{\mathbb{R}^n} f(t)[a.\nabla g(t)]d^n t = \left[\int_{\mathbb{R}^{n-1}} f(t)g(t)d^{n-1}t\right]_{a.t=-\infty}^{a.t=\infty} - \int_{\mathbb{R}^n}[a.\nabla f(t)]g(t)d^n t. \tag{47}$$

Theorem 20 (Directional uncertainty principle for WVD-CT). *Let $f, g \in (\mathbb{R}^n, \mathcal{C}_n)$, such that $h_{f,t}(\tau) = f(t+\frac{\tau}{2})\bar{g}(t-\frac{\tau}{2})$ having the Clifford–Fourier transform $\mathcal{F}\{h_t(\tau)\}(\omega)$. Suppose that $\int_{\mathbb{R}^n} \|h_t(\tau)\|^2 d^n\tau = H < \infty$; then, for any arbitrary constant vectors a, b, the following inequality holds:*

$$\int_{\mathbb{R}^n}(a.\tau)^2\|h_t(\tau)\|^2 d^n\tau.\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}(b.\omega)^2\|\mathcal{F}\{h_t(\tau)\}(\omega)\|^2 d^n\omega \geq (a.b)H^2. \tag{48}$$

Proof.

$$\int_{\mathbb{R}^n}(a.\tau)^2\|h_t(\tau)\|^2 d^n\tau.\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}(b.\omega)^2\|\mathcal{F}\{h_t(\tau)\}(\omega)\|^2 d^n\omega$$
$$= \int_{\mathbb{R}^n}(a.\tau)^2\|h_t(\tau)\|^2 d^n\tau.\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\|\mathcal{F}\{b.\nabla h_t(\tau)\}(\omega)\|^2 d^n\omega \qquad using \quad (44)$$
$$= \int_{\mathbb{R}^n}(a.\tau)^2\|h_t(\tau)\|^2 d^n\tau.\int_{\mathbb{R}^n}\|b.\nabla h_t(\tau)\|^2 d^n\tau \qquad using \quad (33)$$
$$\geq \left(\int_{\mathbb{R}^n}(a.\tau)\|h_t(\tau)\|\|b.\nabla h_t(\tau)\|d^n\tau\right)^2$$
$$\geq \left(\int_{\mathbb{R}^n}a.\tau\left|\langle\overline{h_t(\tau)}b.\nabla h_t(\tau)\rangle\right|d^n\tau\right)^2 \qquad using \quad (11)$$
$$\geq \left(\int_{\mathbb{R}^n}a.\tau\langle\overline{h_t(\tau)}b.\nabla h_t(\tau)\rangle d^n\tau\right)^2.$$

We know that
$$(b.\nabla)\|f\|^2 = 2\langle\bar{f}(b.\nabla)f\rangle$$

Therefore, we have

$$\int_{\mathbb{R}^n}(a.\tau)^2\|h_t(\tau)\|^2 d^n\tau.\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}(b.\omega)^2\|\mathcal{F}\{h_t(\tau)\}(\omega)\|^2 d^n\omega$$
$$\geq \left(\int_{\mathbb{R}^n}a.\tau\frac{1}{2}b.\nabla\|h_t(\tau)\|^2 d^n\tau\right)^2$$
$$\geq \frac{1}{4}\left(\left[\int_{\mathbb{R}^n}a.\tau\|h_t(\tau)\|^2 d^{n-1}\tau-\right]_{b.x=-\infty}^{b.x=-\infty} - \int_{\mathbb{R}^n}(b.\nabla)(a.\tau)\|h_t(\tau)\|d^n\tau\right)^2 \quad using \quad (47)$$
$$= \frac{1}{4}\left(0 - a.b\int_{\mathbb{R}^n}\|h_t(\tau)\|^2 d^n\tau\right)^2.$$
$$= (a.b)H^2$$

which proves (48). This completes the proof. □

Corollary 3. *If $b = \pm a$, that is, $b \parallel a$ with $a^2 = 1$, then the above theorem gives the following result:*

$$\int_{\mathbb{R}^n}(a.\tau)^2\|h_t(\tau)\|^2 d^n\tau.\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}(a.\omega)^2\|\mathcal{F}\{h_t(\tau)\}(\omega)\|^2 d^n\omega \geq \frac{1}{4}H^2.$$

Remark 2. *Equality holds for the Gaussian multivector-valued function*

$$h_t(\tau) = C_0 e^{-k\tau^2}. \tag{49}$$

where $C_0 \in \mathcal{C}_n$ is an arbitrary constant multivector, $0 < k \in \mathbb{R}$.

Therefore, we have from (49)

$$a.\nabla h_t(\tau) = -2ka.\tau h_t(\tau).$$

Theorem 21. *For $a.b = 0$, that is, $a \perp b$, we have*

$$\int_{\mathbb{R}^n}(a.\tau)^2\|h_t(\tau)\|^2 d^n\tau.\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}(\omega)^2\|\mathcal{F}\{h_t(\tau)\}(\omega)\|^2 d^n\omega \geq 0$$

where

$$h_t(\tau) = f(t+\frac{\tau}{2})\bar{g}(t-\frac{\tau}{2}).$$

Theorem 22. *With same assumptions as in theorem (48), we obtain the following result:*

$$\int_{\mathbb{R}^n} (\tau)^2 \|h_t(\tau)\|^2 d^n\tau . \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (\omega)^2 \|\mathcal{F}\{h_t(\tau)\}(\omega)\|^2 d^n\omega \geq \frac{n}{4} H^2.$$

5. Conclusions

The Wigner–Ville distribution has been associated with the one-sided Clifford–Fourier transform over $\mathbb{R}^n$, $n = 3(\text{mod } 4)$, and some fundamental properties of WVD-CFT have been established, such as non- linearity, the shift property, dilation, the vector differential, the vector derivative, and powers of $\tau \in \mathbb{R}^n$. Additionally, some important theorems about WVD-CFT have been formulated, which include the Parseval theorem, the convolution theorem, Moyals formula, and the reconstruction formula. Finally, the directional uncertainty principle associated with WVD-CFT has been derived.

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