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An Algebraic Approach to the Δ_h-Frobenius–Genocchi–Appell Polynomials

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Abstract: In recent years, the generating function of mixed-type special polynomials has received growing interest in several fields of applied sciences and physics. This article intends to study a new class of polynomials, called the Δ_h-Frobenius–Genocchi–Appell polynomials. The generating function of Δ_h-Frobenius–Genocchi–Appell polynomials is constructed and some of their fundamental properties are studied. By making use of this generating function, we investigate some novel and interesting results, such as recurrence relations, explicit representations, and implicit formulas for the Δ_h-Frobenius–Genocchi–Appell polynomials. The quasi-monomiality and determinant form for these polynomials are established. The Δ_h-Genocchi–Appell polynomials are explored as a special case and several results for Δ_h-Genocchi–Appell polynomials are also obtained.

Keywords: Frobenius–Genocchi polynomials; Δ_h–Appell polynomials; quasi-monomiality; determinant form

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1. Introduction and Preliminaries

Special functions, equations, and integers are intensively studied in many disciplines of mathematics, physics, and engineering. The Appell equations and numerals in particular are commonly employed in the creation of fundamental and applied mathematics pertaining to approximation theories, interpolation issues, and quadrature rules (see [1–4]). Many authors have explored several Appell polynomial extensions [5–9]. A new variety of the Appell polynomials known as the Δ_h-Appell polynomials was introduced in [10] by employing the traditional finite difference operator Δ_h. Due to their exceptional usefulness, these Δ_h-Appell polynomials have received a great deal of attention in physics as well as in statistics.

These Δ_h-Appell polynomial are represented as

$$\mathcal{J}_n^h(v) := \mathcal{J}_n(v), \quad n \in \mathbb{N}_0 \tag{1}$$

and defined by

$$\Delta_h\{\mathcal{J}_n(v)\} = nh\mathcal{J}_{n-1}(v), \quad n \in \mathbb{N}, \tag{2}$$

where Δ_h, being f.d.o., is given as [11]

$$\Delta_h[g](v) = g(v+h) - g(v). \tag{3}$$

The Δ_h-Appell polynomials $\mathcal{J}_n(v)$ are specified by the generating expression [10] as follows:

$$\mathcal{J}(t)(1+ht)^{\frac{v}{h}} = \sum_{n=0}^{\infty} \mathcal{J}_n(v)\frac{t^n}{n!}, \tag{4}$$

where

$$\mathcal{J}(t) = \sum_{n=0}^{\infty} \mathcal{J}_{n,h}\frac{t^n}{n!}, \qquad \mathcal{J}_{0,h} \neq 0. \tag{5}$$

The Frobenius–Genocchi equations having order r, $\mathfrak{F}^{[r]}n(v|u)$, are specified by [12]

$$\sum_{n=0}^{\infty} \mathfrak{F}_n^{[r]}(v|u)\,\frac{t^n}{n!} = \left(\frac{(1-u)t}{e^t-u}\right)^r e^{vt} \qquad (u \in \mathbb{C}\setminus\{1\}), \tag{6}$$

for $u \in \mathbb{C}$ with $u \neq 1$ and $n \in \mathbb{Z}^+$.

For polynomials $\mathfrak{F}_n^{[r]}(v|u)$, numerous characterizations, properties, and identities can be found in [13–16]. Taking $v = 0$ in Equation (6), we obtain the corresponding Frobenius–Genocchi numbers $\mathfrak{F}_n^{[r]}(u)$ of order r:

$$\mathfrak{F}_n^{[r]}(u) := \mathfrak{F}_n^{[r]}(0|u)$$

and these numbers $\mathfrak{F}_n^{[r]}(u)$ lead us to give the recurrence relation

$$(\mathfrak{F}(u)+1)^n - \mathfrak{F}_n(u) = ((1-u))\delta_{n,0} \quad \text{and} \quad \mathfrak{F}_0(u) = 1 \tag{7}$$

where the Kronecker delta is denoted by $\delta_{n,k}$.

Moreover, the equations $\mathfrak{F}_n^{[r]}(v|u)$ are stated recursively by the numbers $\mathfrak{F}_n^{[r]}(u)$ as

$$\sum_{k=0}^{n}\binom{n}{k}\mathfrak{F}_k^{[r]}(u)v^{n-k} \qquad (n \geq 0) = \mathfrak{F}_n^{[r]}(v|u). \tag{8}$$

Remark 1. *Taking $u = -1$ and $r = 1$ in the generating Equation (6), the polynomials $\mathfrak{F}_n^{[r]}(v|u)$ reduce to the $G_n(v)$ polynomials*

$$\mathfrak{F}_n^{[1]}(v|-1) = G_n(v),$$

which are stated as

$$\frac{2t}{e^t+1}e^{vt} = \sum_{n=0}^{\infty} G_n(v)\frac{t^n}{n!}. \tag{9}$$

Now, we recall basic definitions that are mandatory throughout this study.

Definition 1. *The expressions stated in [17]*

$$\sum_{n=m}^{\infty} \mathfrak{S}_1(n,m)\frac{t^n}{n!} = \frac{(\log(1+t))^m}{m!}, \tag{10}$$

are called Stirling integers $\mathfrak{S}_1(n,m)$ of the first kind.

Definition 2. *The expression stated by*

$$(v|\lambda)_n = \Pi_{k=0}^{n-1}(v-\lambda k), \tag{11}$$

is called a simplified descending factorial $(v|\lambda)_n$ with incremental λ, established for positive integer n, with the notion $(v|\lambda)_0 = 1$.

It follows that

$$(v|\lambda)_n = \sum_{k=0}^{n} \mathfrak{S}_1(n,k)\lambda^{n-k}v^k. \tag{12}$$

From the Binomial theorem, we have

$$(1+\lambda t)^{\frac{v}{\lambda}} = \sum_{n=0}^{\infty} (v|\lambda)_n \frac{t^n}{n!}. \tag{13}$$

We define the latest subclass of the Δ_h-special functions and prove numerous identities relating to these polynomials, which are inspired by the work in the direction of obtaining Δ_h-special functions.

The rest of the paper is organized in the following manner. Section 2 introduces and establishes various fresh identities for Δ_h-Frobenius–Genocchi polynomials, as well as their hybrid forms. Section 3 provides the Δ_h-Frobenius–Genocchi–Appell polynomials' quasi-monomiality and determinant forms. As a specific instance of Δ_h-Frobenius–Genocchi–Appell polynomials, Δ_h-Genocchi–Appell polynomials are introduced in Section 4, along with relevant findings. Finally, concluding observations and remarks are provided in Section 5.

2. Δ_h-Frobenius–Genocchi–Appell Equations

In this section, the Δ_h-Frobenius–Genocchi equations are explained before providing the Δ_h-Frobenius–Genocchi–Appell polynomials' generating function. Additionally, several novel identities for these polynomials are obtained. We provide the definitions below.

Definition 3. *For $v \in \mathbb{R}$, $u \in \mathbb{C}$ with $u \neq 1$ and $n \in \mathbb{Z}^+$. The expression stated by*

$$\left(\frac{(1-u)\frac{log(1+ht)}{h}}{(1+ht)^{\frac{1}{h}} - u} \right)^r (1+ht)^{\frac{v}{h}} = \sum_{n=0}^{\infty} \mathfrak{F}_n^{[r,h]}(v|u)\frac{t^n}{n!}, \tag{14}$$

defines the generating expression for the Δ_h-Frobenius–Genocchi polynomials, represented by $\mathfrak{F}_n^{[r,h]}(v|u)$ of order r. This, on taking $v = 0$, gives the corresponding numbers $\mathfrak{F}_n^{[r,h]}(u)$ of order r listed as

$$\left(\frac{(1-u)\frac{log(1+ht)}{h}}{(1+ht)^{\frac{1}{h}} - u} \right)^r = \sum_{n=0}^{\infty} \mathfrak{F}_n^{[r,h]}(u)\frac{t^n}{n!}. \tag{15}$$

Remark 2. *Taking $h \to 0$ in Equation (14), we obtain*

$$\lim_{h\to 0} \mathfrak{F}_n^{[r,h]}(v|u) = \mathfrak{F}_n^{[r]}(v|u), \quad n \geq 0, \tag{16}$$

where $\mathfrak{F}_n^{[r]}(v|u)$ are the Frobenius–Genocchi polynomials of order r mentioned in (6).

The development of hybrid forms of mathematical physics' special functions has seen great strides. A more recent method is to introduce hybridized polynomial forms and describe their characteristics using generating functions. Hybrid special equations are noteworthy because they have important properties, such as explicit relations, differential and difference expressions, summation formulae, symmetrical and convolutional identities, and determinant methods. The properties of hybrid distinct equations could be used to resolve new difficulties in a range of scientific and technological domains.

In view of Equations (4) and (14), we define the Δ_h-Frobenius–Genocchi–Appell polynomials (Δ_h FGAP), denoted by ${}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(v|u)$ of order r, as

Definition 4. *Let $v \in \mathbb{R}$; $u \in \mathbb{C}$ with $u \neq 1$ and $n \in \mathbb{Z}^+$. The Δ_h FGAP ${}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(v|u)$ having order r are given by the below-mentioned generative equation:*

$$\mathcal{J}(t)\left(\frac{(1-u)\frac{log(1+ht)}{h}}{(1+ht)^{\frac{1}{h}}-u}\right)^r(1+ht)^{\frac{v}{h}} = \sum_{n=0}^{\infty} {}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(v|u)\frac{t^n}{n!}. \tag{17}$$

In consideration of $v = 0$, Equation (17) gives the corresponding Δ_h-Frobenius–Genocchi–Appell numbers ${}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(u)$ of order r, defined as

$$\mathcal{J}(t)\left(\frac{(1-u)\frac{log(1+ht)}{h}}{(1+ht)^{\frac{1}{h}}-u}\right)^r = \sum_{n=0}^{\infty} {}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(u)\frac{t^n}{n!}, \tag{18}$$

where $\mathcal{J}(t)$ is the same as in Equation (5).

Theorem 1. *For any integral $n \geq 1$, the underlying recurrence condition for Δ_h FGAP ${}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(v|u)$ holds true:*

$$\begin{aligned} {}_{\mathfrak{F}}\mathcal{J}_{n+1}^{[r,h]}(v|u) &= \left(v + r\frac{h}{log(1+ht)}\right){}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(v-h|u) \\ &\quad - \frac{r}{(1+ht)^{\frac{1}{h}}-u}{}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(v+1-h|u) + \sum_{k=0}^{n}\binom{n}{k}\beta_{k,h}{}_{\mathfrak{F}}\mathcal{J}_{n-k}^{[r,h]}(v|u) \end{aligned} \tag{19}$$

Proof. By taking derivatives of (17) with respect to t, we have

$$\begin{aligned} &v\,\mathcal{J}(t)\left(\frac{(1-u)\frac{log(1+ht)}{h}}{(1+ht)^{\frac{1}{h}}-u}\right)^r(1+ht)^{\frac{v}{h}-1} + \frac{\mathcal{J}'(t)}{\mathcal{J}(t)}\mathcal{J}(t)\left(\frac{(1-u)\frac{log(1+ht)}{h}}{(1+ht)^{\frac{1}{h}}-u}\right)^r(1+ht)^{\frac{v}{h}} \\ &\quad - \frac{r}{(1+ht)^{\frac{1}{h}}-u}\mathcal{J}(t)\left(\frac{(1-u)\frac{log(1+ht)}{h}}{(1+ht)^{\frac{1}{h}}-u}\right)^r(1+ht)^{\frac{v+1}{h}-1} \\ &\quad + r\frac{h}{log(1+ht)}\mathcal{J}(t)\left(\frac{(1-u)\frac{log(1+ht)}{h}}{(1+ht)^{\frac{1}{h}}-u}\right)^r(1+ht)^{\frac{v}{h}-1} = \sum_{n=0}^{\infty} n\,{}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(v|u)\frac{t^{n-1}}{n!}. \end{aligned} \tag{20}$$

Taking $\frac{\mathcal{J}'(t)}{\mathcal{J}(t)} = \sum\limits_{k=0}^{\infty}\beta_{k,h}\frac{t^k}{k!}$ and applying Equation (17), it was found that

$$\begin{aligned} &\left(v + r\frac{h}{log(1+ht)}\right)\sum_{n=0}^{\infty}{}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(v-h|u)\frac{t^n}{n!} + \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\beta_{k,h}\,{}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(v|u)\frac{t^{n+k}}{n!\,k!} \\ &\quad - \frac{r}{(1+ht)^{\frac{1}{h}}-u}\sum_{n=0}^{\infty}{}_{\mathfrak{F}}\mathcal{J}_n^{[r+1,h]}(v+1-h|u)\frac{t^n}{n!} = \sum_{n=0}^{\infty}{}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(v|u)\frac{t^{n-1}}{n!}. \end{aligned} \tag{21}$$

Applying the Cauchy product rule and then equating the coefficients of t on both sides of Equation (21), assertion (19) is proven. □

Theorem 2. *For the Δ_h FGAP ${}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(v|u)$, the following implicit formulae hold true.*

$$(a)\quad {}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(v|u) = \sum_{k=0}^{n}\binom{n}{k}(z|h)_k{}_{\mathfrak{F}}\mathcal{J}_{n-k}^{[r,h]}(v-z|u). \tag{22}$$

$$(b)\quad {}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(v+z|u) = \sum_{k=0}^{n}\binom{n}{k}(z|h)_k{}_{\mathfrak{F}}\mathcal{J}_{n-k}^{[r,h]}(v|u). \tag{23}$$

$$(c)\quad {}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(v+z|u) = \sum_{l=0}^{n}\sum_{k=l}^{\infty}\binom{n}{l}\mathfrak{S}_1(l,k)z^k h^{l-k}{}_{\mathfrak{F}}\mathcal{J}_{n-l}^{[r,h]}(v|u). \tag{24}$$

Proof. (a) The generating Equation (17) can be written as

$$\mathcal{J}(t)\left(\frac{(1-u)\frac{log(1+ht)}{h}}{(1+ht)^{\frac{1}{h}}-u}\right)^{r}(1+ht)^{\frac{v-z}{h}}=\sum_{n=0}^{\infty}{}_{\mathfrak{F}}\mathcal{J}_{n}^{[r,h]}(v-z|u)\frac{t^{n}}{n!}.$$

Consequently,

$$\mathcal{J}(t)\left(\frac{(1-u)\frac{log(1+ht)}{h}}{(1+ht)^{\frac{1}{h}}-u}\right)^{r}(1+ht)^{\frac{v}{h}}=(1+ht)^{\frac{z}{h}}\sum_{n=0}^{\infty}{}_{\mathfrak{F}}\mathcal{J}_{n}^{[r,h]}(v-z|u)\frac{t^{n}}{n!} \tag{25}$$

Now, using relation (13) and the generating Equation (17), we have

$$\sum_{n=0}^{\infty}{}_{\mathfrak{F}}\mathcal{J}_{n}^{[r,h]}(v|u)\frac{t^{n}}{n!}=\left(\sum_{k=0}^{\infty}(z|h)_{k}\frac{t^{k}}{k!}\right)\sum_{n=0}^{\infty}{}_{\mathfrak{F}}\mathcal{J}_{n}^{[r,h]}(v-z|u)\frac{t^{n}}{n!}. \tag{26}$$

Assertion (22) is obtained by applying the Cauchy product rule on the b/s of (26) by subsequently comparing the coefficient of t.

(b) The generating Equation (17) can be written as

$$\mathcal{J}(t)\left(\frac{(1-u)\frac{log(1+ht)}{h}}{(1+ht)^{\frac{1}{h}}-u}\right)^{r}(1+ht)^{\frac{v+z}{h}}=\sum_{n=0}^{\infty}{}_{\mathfrak{F}}\mathcal{J}_{n}^{[r,h]}(v+z|u)\frac{t^{n}}{n!}, \tag{27}$$

Now, using relation (13), we have

$$\left(\sum_{n=0}^{\infty}{}_{\mathfrak{F}}\mathcal{J}_{n}^{[r,h]}(v|u)\frac{t^{n}}{n!}\right)\left(\sum_{k=0}^{\infty}(z|h)_{k}\frac{t^{k}}{k!}\right)=\sum_{n=0}^{\infty}{}_{\mathfrak{F}}\mathcal{J}_{n}^{[r,h]}(v+z|u)\frac{t^{n}}{n!}. \tag{28}$$

Assertion (23) is obtained by applying the C.P. rule on the b/s of (28) by subsequently comparing the coefficient of t.

(c) Equation (27) can be written as

$$\left(\sum_{n=0}^{\infty}{}_{\mathfrak{F}}\mathcal{J}_{n}^{[r,h]}(v|u)\frac{t^{n}}{n!}\right)\left(\sum_{k=0}^{\infty}\left(\frac{z}{h}\right)^{k}\frac{(\log(1+ht))^{k}}{k!}\right)=\sum_{n=0}^{\infty}{}_{\mathfrak{F}}\mathcal{J}_{n}^{[r,h]}(v+z|u)\frac{t^{n}}{n!},$$

Now, using Equation (10), we have

$$\left(\sum_{n=0}^{\infty}{}_{\mathfrak{F}}\mathcal{J}_{n}^{[r,h]}(v|u)\frac{t^{n}}{n!}\right)\left(\sum_{k=0}^{\infty}\left(\frac{z}{h}\right)^{k}\sum_{l=k}^{\infty}\mathfrak{S}_{1}(l,k)\frac{(ht)^{l}}{l!}\right)=\sum_{n=0}^{\infty}{}_{\mathfrak{F}}\mathcal{J}_{n}^{[r,h]}(v+z|u)\frac{t^{n}}{n!}. \tag{29}$$

Consequently,

$$\sum_{n,l=0}^{\infty}\sum_{k=l}^{\infty}{}_{\mathfrak{F}}\mathcal{J}_{n}(v|u)\left(\frac{z}{h}\right)^{k}\mathfrak{S}_{1}(l,k)h^{l}\frac{t^{n+l}}{n!\,l!}=\sum_{n=0}^{\infty}{}_{\mathfrak{F}}\mathcal{J}_{n}^{[r,h]}(v+z|u)\frac{t^{n}}{n!}. \tag{30}$$

Substituting n by $n-l$ in Equation (30) and comparing the coefficients t, the assertion (24) follows. □

Theorem 3. *The explicit formula for Δ_h FGAP ${}_{\mathfrak{F}}\mathcal{J}_{n}^{[r,h]}(v|u)$ in the context of the Stirling number of the first kind $\mathfrak{S}_1(n,m)$ is as follows.*

$$\begin{aligned}{}_{\mathfrak{F}}\mathcal{J}_{k+1}^{[r,h]}(v|u)=\sum_{n=0}^{k}\sum_{k=0}^{n}\binom{n}{k}\Bigg\{&\left(v(v-h)^{m}-\frac{r}{1-u}(v+1-h)^{m}\right){}_{\mathfrak{F}}\mathcal{J}_{k-n}^{[r,h]}(u)\\&-\sum_{p=0}^{k-n}\binom{k-n}{p}v^{m}\beta_{p,h}\,{}_{\mathfrak{F}}\mathcal{J}_{k-n-p}^{[r,h]}(u)\Bigg\}h^{n-m}\mathfrak{S}_{1}(n,m).\end{aligned} \tag{31}$$

Proof. We can rewrite Equation (20) as

$$v\,\mathcal{J}(t)\left(\frac{(1-u)\frac{log(1+ht)}{h}}{(1+ht)^{\frac{1}{h}}-u}\right)^{r} e^{\frac{v-h}{h}\log(1+ht)}+\frac{\mathcal{J}'(t)}{\mathcal{J}(t)}\mathcal{J}(t)\left(\frac{(1-u)\frac{log(1+ht)}{h}}{(1+ht)^{\frac{1}{h}}-u}\right)^{r} e^{\frac{v}{h}\log(1+ht)}$$
$$-\frac{r}{(1+ht)^{\frac{1}{h}}-u}\mathcal{J}(t)\left(\frac{(1-u)\frac{log(1+ht)}{h}}{(1+ht)^{\frac{1}{h}}-u}\right)^{r} e^{\frac{v+1-h}{h}}(1+ht)$$
$$+r\frac{h}{log(1+ht)}\mathcal{J}(t)\left(\frac{(1-u)\frac{log(1+ht)}{h}}{(1+ht)^{\frac{1}{h}}-u}\right)^{r} e^{\frac{v-h}{h}\log(1+ht)}=\sum_{k=0}^{\infty}k\,{}_{\mathfrak{F}}\mathcal{J}_k^{[r,h]}(v|u)\frac{t^{k-1}}{k!},$$

which can further be simplified as

$$\begin{aligned}&\Big(v+r\frac{h}{log(1+ht)}\Big)\mathcal{J}(t)\left(\frac{(1-u)\frac{log(1+ht)}{h}}{(1+ht)^{\frac{1}{h}}-u}\right)^{r} e^{\frac{v-h}{h}\log(1+ht)}\\&+\frac{\mathcal{J}'(t)}{\mathcal{J}(t)}\mathcal{J}(t)\left(\frac{(1-u)\frac{log(1+ht)}{h}}{(1+ht)^{\frac{1}{h}}-u}\right)^{r} e^{\frac{v}{h}\log(1+ht)}\\&-\frac{r}{(1+ht)^{\frac{1}{h}}-u}\mathcal{J}(t)\left(\frac{(1-u)\frac{log(1+ht)}{h}}{(1+ht)^{\frac{1}{h}}-u}\right)^{r} e^{\frac{v+1-h}{h}}(1+ht)\\&=\sum_{k=0}^{\infty}k\,{}_{\mathfrak{F}}\mathcal{J}_k^{[r,h]}(v|u)\frac{t^{k-1}}{k!}.\end{aligned}\tag{32}$$

Expanding the exponential and then using Equations (17) and (10) and taking $\frac{\mathcal{J}'(t)}{\mathcal{J}(t)}=\sum_{p=0}^{\infty}\beta_{p,h}\frac{t^p}{p!}$, it follows from the modification and resulting equation that

$$\begin{aligned}&\Big(v+r\frac{h}{log(1+ht)}\Big)\sum_{k=0}^{\infty}{}_{\mathfrak{F}}\mathcal{J}_k^{[r,h]}(u)\frac{t^k}{k!}\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\left(\frac{v-h}{h}\right)^m\mathfrak{S}_1(n,m)\right)\frac{h^nt^n}{n!}\\&+\sum_{k=0}^{\infty}\sum_{p=0}^{k}\beta_{p,h}\,{}_{\mathfrak{F}}\mathcal{J}_{k-p}^{[r,h]}(u)\frac{t^k}{p!\,(k-p)!}\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\left(\frac{v}{h}\right)^m\mathfrak{S}_1(n,m)\right)\frac{h^nt^n}{n!}\\&-\frac{r}{(1+ht)^{\frac{1}{h}}-u}\sum_{k=0}^{\infty}{}_{\mathfrak{F}}\mathcal{J}_k^{[r,h]}(u)\frac{t^k}{k!}\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\left(\frac{v+1-h}{h}\right)^m\mathfrak{S}_1(n,m)\right)\frac{h^nt^n}{n!}=\sum_{k=0}^{\infty}{}_{\mathfrak{F}}\mathcal{J}_{k+1}^{[r,h]}(v|u)\frac{t^k}{k!}.\end{aligned}\tag{33}$$

Taking the coefficients of identical powers of t in Equation (33) and subsequently equating and exchanging both sides yields assertion (31). □

Theorem 4. *The Δ_h FGAP ${}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(v|u)$ could be represented in the form of Δ_h-Frobenius–Genocchi polynomials $\mathfrak{F}_n^{[r,h]}(v|u)$ and Δ_h-Appell polynomials $\mathcal{J}_k(v)$, respectively, by the following explicit representations:*

$$(i)\quad {}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(v|u)=\sum_{k=0}^{n}\binom{n}{k}\alpha_{k,h}\,\mathfrak{F}_{n-k}^{[r,h]}(v|u).\tag{34}$$

$$(ii)\quad {}_{\mathfrak{F}}\mathcal{J}_n^{[1,-h]}(-v|u)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\alpha_{n-k,h}\,\mathfrak{F}_k^{[1,h]}(v+1|u^{-1}).\tag{35}$$

$$(iii)\quad {}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(v|u)=\sum_{k=0}^{n}\binom{n}{k}\mathfrak{F}_{n-k}^{[r,h]}(u)\,\mathcal{J}_k(v).\tag{36}$$

Proof. (i) Inserting Equations (5) and (14) in the l.h.s. of Equation (17), we obtain

$$\sum_{n=0}^{\infty} {}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(v|u)\frac{t^n}{n!}\sum_{k=0}^{\infty}\alpha_{k,h}\frac{t^k}{k!} = \sum_{n=0}^{\infty} {}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(v|u)\frac{t^n}{n!}. \tag{37}$$

Assertion (34) is the result of using the Cauchy product rule on the l.h.s. of the previous Equation (26) and subsequently replacing n with $n-k$.

(ii) In consideration of $h \to -h$, $v \to -v$ and $r = 1$, Equation (17) yields

$$\sum_{n=0}^{\infty} {}_{\mathfrak{F}}\mathcal{J}_n^{[1,-h]}(-v|u)\frac{t^n}{n!} = \mathcal{J}(t)\left(\frac{(1-u)\frac{log(1-ht)}{-h}}{(1-ht)^{-\frac{1}{h}}-u}\right)(1-ht)^{\frac{v}{h}}. \tag{38}$$

On simplification, we obtain

$$\sum_{n=0}^{\infty} {}_{\mathfrak{F}}\mathcal{J}_n^{[1,-h]}(-v|u)\frac{t^n}{n!} = \mathcal{J}(t)\left(\frac{(1-u^{-1})\frac{log(1-ht)}{-h}}{(1-ht)^{\frac{1}{h}}-u^{-1}}\right)(1-ht)^{\frac{v+1}{h}}, \tag{39}$$

Assertion (35) is the result of using Equations (14) and (5) subsequent to the reordering of a sequence and comparison of the coefficient.

(iii) Inserting expressions (4) and (15) in the left-hand side of expression (17),

$$\sum_{n=0}^{\infty} {}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(u)\frac{t^n}{n!}\sum_{k=0}^{\infty}\mathcal{J}_k(v)\frac{t^k}{k!} = \sum_{n=0}^{\infty} {}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(v|u)\frac{t^n}{n!}. \tag{40}$$

Assertion (36) is the result of employing the Cauchy product rule on the l.h.s. of the previous expression by replacing n with $n-k$. □

Theorem 5. *For Δ_h FGAP ${}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(v|u)$, we have the following identities:*

$$(a)\quad {}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(v|u) = \sum_{k=0}^{n}\binom{n}{k}(v|h)_k\,{}_{\mathfrak{F}}\mathcal{J}_{n-k}^{[r,h]}(u). \tag{41}$$

$$(b)\quad {}_{\mathfrak{F}}\mathcal{J}_n^{[1,h]}(v+1|u) - u\,{}_{\mathfrak{F}}\mathcal{J}_n^{[1,h]}(u) = (1-u)\sum_{k=0}^{n}\binom{n}{k}{}_{\mathfrak{F}}\mathcal{J}_{n-k}^{[1,h]}(u)\,(v|h)_k. \tag{42}$$

$$(c)\quad {}_{\mathfrak{F}}\mathcal{J}_n^{[h]}(1|u) - u\,{}_{\mathfrak{F}}\mathcal{J}_n^{[h]}(u) = (1-u)\sum_{k=0}^{n}\binom{n}{k}{}_{\mathfrak{F}}\mathcal{J}_{n-k}^{[1,h]}(u). \tag{43}$$

Proof. (a) In the context of (13), deriving function (17) may be expressed as

$$\left(\sum_{n=0}^{\infty} {}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(u)\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty}(v|h)_k\frac{t^k}{k!}\right) = \sum_{n=0}^{\infty} {}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(v|u)\frac{t^n}{n!}. \tag{44}$$

Assertion (41) is obtained by applying the C.P. on the b/s of the previous equation by subsequently comparing the coefficient of t.

(b) Taking $r = 1$ in Equation (17), we have

$$\begin{aligned}\sum_{n=0}^{\infty} {}_{\mathfrak{F}}\mathcal{J}_n^{[1,h]}(v+1|u)\frac{t^n}{n!} - u\sum_{n=0}^{\infty} {}_{\mathfrak{F}}\mathcal{J}_n^{[1,h]}(v|u)\frac{t^n}{n!} =& \mathcal{J}(t)\left(\frac{(1-u)\frac{log(1+ht)}{h}}{(1+ht)^{\frac{1}{h}}-u}\right)\Big(1+ht\Big)^{\frac{v+1}{h}}\\ &- u\mathcal{J}(t)\left(\frac{(1-u)\frac{log(1+ht)}{h}}{(1+ht)^{\frac{1}{h}}-u}\right)\Big(1+ht\Big)^{\frac{v}{h}}\\ &= (1-u)\mathcal{J}(t)\left(\frac{(1-u)\frac{log(1+ht)}{h}}{(1+ht)^{\frac{1}{h}}-u}\right)\Big(1+ht\Big)^{\frac{v}{h}}.\end{aligned} \tag{45}$$

Using relations (5) and (18) in Equation (45), we obtain

$$\sum_{n=0}^{\infty} {}_{\mathfrak{F}}\mathcal{J}_n^{[1,h]}(v+1|u)\frac{t^n}{n!} - u\sum_{n=0}^{\infty} {}_{\mathfrak{F}}\mathcal{J}_n^{[1,h]}(v|u)\frac{t^n}{n!} = (1-u)\sum_{n=0}^{\infty} {}_{\mathfrak{F}}\mathcal{J}_n^{[1,h]}(u)\frac{t^n}{n!}\sum_{k=0}^{\infty}(v|h)_k\frac{t^k}{k!}. \tag{46}$$

Assertion (42) is proven by using the C.P. rule followed by equating coefficients of identical powers in the resultant equation.

(c) Result (43) can be obtained by taking $v = 0$ in relation (42). □

In the next section, the quasi-monomiality and determinant form for the Δ_h FGAP ${}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(v|u)$ are established.

3. Quasi-Monomiality and Determinant Form

Dattoli [18] introduced and thoroughly examined the idea of quasi-monomiality. Finding the multiplicative and derivative operators is the major goal here. Additionally, we establish the following conclusion to frame the Δ_h FGAP ${}_{\mathfrak{F}}\mathcal{J}^{[r,h]}n(v|u)$ order r within the monomiality principle's framework.

Theorem 6. *With respect to the Δ_h-${}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(v|u)$ polynomials, the following multiplying and differential operators exhibit quasi-monomial features:*

$$\hat{M}_{F^{(r)}\mathcal{J}} = \left(\frac{\mathcal{J}'\left(\frac{e^{hD_v}-1}{h}\right)}{\mathcal{J}\left(\frac{e^{hD_v}-1}{h}\right)} + \frac{v+r\frac{1}{D_v}}{e^{hD_v}} - \frac{r}{e^{hD_v}\left(1-\frac{u}{e^{D_v}}\right)}\right) \tag{47}$$

and

$$\hat{P}_{F^{(r)}\mathcal{J}} = \frac{e^{hD_v}-1}{h}. \tag{48}$$

Proof. Contemplate the identity

$$\frac{1}{h}\log(1+ht)\left\{e^{v\log(1+ht)^{\frac{1}{h}}}\right\} = D_v\left\{e^{v\log(1+ht)^{\frac{1}{h}}}\right\}. \tag{49}$$

We have

$$t\left\{e^{v\log(1+ht)^{\frac{1}{h}}}\right\} = \frac{e^{hD_v}-1}{h}\left\{e^{v\log(1+ht)^{\frac{1}{h}}}\right\}. \tag{50}$$

When the generative function (17) is partly differentiated with regard to t, it implies that

$$\begin{aligned}&\left[\left(v+r\frac{h}{log(1+ht)}\right)(1+ht)^{-1}+\frac{\mathcal{J}'(t)}{\mathcal{J}(t)}-\frac{r}{(1+ht)^{\frac{1}{h}}-u}(1+ht)^{\frac{1}{h}-1}\right.\\&\left.+r\frac{h}{log(1+ht)}(1+ht)^{-1}\right]\mathcal{J}(t)\left(\frac{(1-u)\frac{log(1+ht)}{h}}{(1+ht)^{\frac{1}{h}}-u}\right)^r(1+ht)^{\frac{v}{h}} = \sum_{n=0}^{\infty} {}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(v|u)\frac{t^{n-1}}{(n-1)!}.\end{aligned} \tag{51}$$

Hence, after reordering the summation and using generative function (17) and identity (50) in the left-hand side of the resultant expression, we obtain

$$\sum_{n=0}^{\infty}\left(\frac{\mathcal{J}'\left(\frac{e^{hD_v}-1}{h}\right)}{\mathcal{J}\left(\frac{e^{hD_v}-1}{h}\right)} + \frac{v+r\frac{1}{D_v}}{e^{hD_v}} - \frac{r}{e^{hD_v}\left(1-\frac{u}{e^{D_v}}\right)}\right)\left({}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(v|u)\frac{t^n}{n!}\right) = \sum_{n=0}^{\infty} {}_{\mathfrak{F}}\mathcal{J}_{n+1}^{[r,h]}(v|u)\frac{t^n}{n!}. \tag{52}$$

Owing to the monomiality principle's expression $\hat{M}\{p_n(v)\} = p_{n+1}(v)$ and the coefficients of the same powers of t on both sides of Equation (52), statement (47) is proven.

In view of identity (50), we have

$$t\left\{\mathcal{J}(t)\left(\frac{(1-u)\frac{\log(1+ht)}{h}}{(1+ht)^{\frac{1}{h}}-u}\right)^r(1+ht)^{\frac{v}{h}}\right\}=\frac{e^{hD_v}-1}{h}\left\{\mathcal{J}(t)\left(\frac{(1-u)\frac{\log(1+ht)}{h}}{(1+ht)^{\frac{1}{h}}-u}\right)^r(1+ht)^{\frac{v}{h}}\right\}. \tag{53}$$

Using the generating Equation (17) on both sides and interchanging the sides, we have

$$\frac{e^{hD_v}-1}{h}\left\{\sum_{n=0}^{\infty}{}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(v|u)\frac{t^n}{n!}\right\}=\sum_{n=1}^{\infty}{}_{\mathfrak{F}}\mathcal{J}_{n-1}^{[r,h]}(v|u)\frac{t^n}{(n-1)!}. \tag{54}$$

Owing to the monomiality principle equation $\hat{P}\{p_n(v)\}=n\,p_{n-1}(v)$ and the comparison of the coefficients having similar powers of t in the left-hand as well as the right-hand sides of expression (54), expression (48) follows. □

Employing Equations (47) and (48) in the monomiality principle's equation $\hat{M}\hat{P}\{p_n(v)\}=n\,p_n(v)$, the following conclusion can be drawn.

Corollary 1. *For the Δ_h FGAP ${}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(v|u)$, we have the following differential equation:*

$$\left(\frac{\mathcal{J}'\left(\frac{e^{hD_v}-1}{h}\right)}{\mathcal{J}\left(\frac{e^{hD_v}-1}{h}\right)}+\frac{v+r\frac{1}{D_v}}{e^{hD_v}}-\frac{r}{e^{hD_v}\left(1-\frac{u}{e^{D_v}}\right)}-n\frac{h}{e^{hD_v}-1}\right){}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(v|u)=0. \tag{55}$$

Theorem 7. *The Δ_h FGAP ${}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(v|u)$ gives rise to the determinant in the following form:*

$${}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(v|u)=\frac{(-1)^n}{(\gamma_{0,h})^{n+1}}\begin{vmatrix}1 & \mathfrak{F}_1^{[r,h]}(v|u) & \mathfrak{F}_2^{[r,h]}(v|u) & \cdots & \mathfrak{F}_{n-1}^{[r,h]}(v|u) & \mathfrak{F}_n^{[r,h]}(v|u)\\ \gamma_{0,h} & \gamma_{1,h} & \gamma_{2,h} & \cdots & \gamma_{n-1,h} & \gamma_{m,h}\\ 0 & \gamma_{0,h} & \binom{2}{1}\gamma_{1,h} & \cdots & \binom{m-1}{1}\gamma_{n-2,h} & \binom{n}{1}\gamma_{n-1,h}\\ 0 & 0 & \gamma_{0,h} & \cdots & \binom{n-1}{2}\gamma_{n-3,h} & \binom{n}{2}\gamma_{n-2,h}\\ . & . & . & \cdots & . & .\\ . & . & . & \cdots & . & .\\ 0 & 0 & 0 & \cdots & \gamma_{0,h} & \binom{m}{m-1}\gamma_{1,h}\end{vmatrix}, \tag{56}$$

where

$$\gamma_{m,h},\ m=0,1,\cdots \textit{are the coefficients of Maclaurin's series of } \frac{1}{\mathcal{J}(t)}.$$

Proof. Multiplying both sides of Equation (17) by $\frac{1}{\mathcal{J}(t)}=\sum_{k=0}^{\infty}\gamma_{m,h}\frac{t^m}{m!}$, we find

$$\sum_{n=0}^{\infty}\mathfrak{F}_n^{[r,h]}(v|u)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\gamma_{m,h}\frac{t^m}{m!}\,{}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(v|u)\frac{t^n}{n!}. \tag{57}$$

Using the Cauchy product rule, we have

$$\mathfrak{F}_n^{[r,h]}(v|u)=\sum_{m=0}^{n}\binom{n}{m}\gamma_{m,h}\,{}_{\mathfrak{F}}\mathcal{J}_{n-m}^{[r,h]}(v|u). \tag{58}$$

The system of m-equations with unknowns ${}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(v|u),\ n=0,1,2,\cdots$. is generated by this equality. Applying Cramer's rule, as well as the understanding that the denominator is the determinant of the lower triangular matrix $(\gamma_{0,h})^{n+1}$, the requisite result may be achieved by transposing the numerator, and then substituting the i-th row with the $(i+1)$-th position for $i=1,2,\cdots,n-1$. □

In the following section, Δ_h-Genocchi–Appell polynomial ${}_G\mathcal{J}_n^{[h]}(v)$ is introduced as a special case of Δ_h FGAP ${}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(v|u)$ of order r.

4. Special Case

In consideration of $u=-1$ and $r=1$, generating Equation (17) gives the generating equation of Δ_h-Genocchi–Appell polynomials ${}_G\mathcal{J}_n^{[h]}(v)$, defined as

$$\mathcal{J}(t)\left(\frac{2^{\frac{log(1+ht)}{h}}}{(1+ht)^{\frac{1}{h}}+1}\right)(1+ht)^{\frac{v}{h}}=\sum_{n=0}^{\infty}{}_G\mathcal{J}_n^{[h]}(v)\frac{t^n}{n!},\tag{59}$$

which for $v=0$ gives the corresponding Δ_h-Genocchi–Appell numbers ${}_G\mathcal{J}_n^{[h]}$, given by

$$\mathcal{J}(t)\left(\frac{2^{\frac{log(1+ht)}{h}}}{(1+ht)^{\frac{1}{h}}+1}\right)=\sum_{n=0}^{\infty}{}_G\mathcal{J}_n^{[h]}\frac{t^n}{n!}.\tag{60}$$

Remark 3. *Taking $\mathcal{J}(t)=1$ in Equation (59), we obtain the generating function of Δ_h-Genocchi polynomials $G_n^{[h]}(v)$:*

$$\left(\frac{2^{\frac{log(1+ht)}{h}}}{(1+ht)^{\frac{1}{h}}+1}\right)(1+ht)^{\frac{v}{h}}=\sum_{n=0}^{\infty}G_n^{[h]}(v)\frac{t^n}{n!}.\tag{61}$$

When $v=0$, $G_n^{[h]}:=G_n^{[h]}(0)$ gives the corresponding Δ_h-Genocchi numbers.

Theorem 8. *The Δ_h-Genocchi–Appell polynomials ${}_G\mathcal{J}_n^{[h]}(v)$ are given by the following determinant form:*

$${}_G\mathcal{J}_n^{[h]}(v)=\frac{(-1)^n}{(\gamma_{0,h})^{n+1}}\begin{vmatrix} 1 & G_1^{[h]}(v) & G_2^{[h]}(v) & \cdots & G_{n-1}^{[h]}(v) & G_n^{[h]}(v) \\ \gamma_{0,h} & \gamma_{1,h} & \gamma_{2,h} & \cdots & \gamma_{n-1,h} & \gamma_{m,h} \\ 0 & \gamma_{0,h} & \binom{2}{1}\gamma_{1,h} & \cdots & \binom{m-1}{1}\gamma_{n-2,h} & \binom{n}{1}\gamma_{n-1,h} \\ 0 & 0 & \gamma_{0,h} & \cdots & \binom{n-1}{2}\gamma_{n-3,h} & \binom{n}{2}\gamma_{n-2,h} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & \gamma_{0,h} & \binom{m}{m-1}\gamma_{1,h} \end{vmatrix},\tag{62}$$

where

$$\gamma_{m,h},\ m=0,1,\cdots \textit{are the coefficients of Maclaurin's series of } \frac{1}{\mathcal{J}(t)}.$$

The other results for the Δ_h-Genocchi–Appell polynomials ${}_G\mathcal{J}_n^{[h]}(v)$ are given in Table 1.

Table 1. Results for ${}_{G}\mathcal{J}_n^{[h]}(v)$.

S. No.	Result	Expression
I.	Multiplicative and derivative operators	$\hat{M}_{G\mathcal{J}} = \left(\frac{j'\left(\frac{e^{hD_v}-1}{h}\right)}{j\left(\frac{e^{hD_v}-1}{h}\right)} + \frac{v+r\frac{1}{D_v}}{e^{hD_v}} - \frac{e^{-hD_v}}{1+e^{-D_v}} \right)$ $\hat{P}_{E\mathcal{J}} = \frac{e^{hD_v}-1}{h}$
II.	Differential equation	$\left(\frac{j'\left(\frac{e^{hD_v}-1}{h}\right)}{j\left(\frac{e^{hD_v}-1}{h}\right)} + \frac{v+r\frac{1}{D_v}}{e^{hD_v}} - \frac{e^{-hD_v}}{1+e^{-D_v}} - n\frac{h}{e^{hD_v-1}} \right) {}_{E}\mathcal{J}_n^{[h]}(v) = 0$
III.	Recurrence relation	${}_{G}\mathcal{J}_{n+1}^{[h]}(v) = (v + r\frac{h}{log(1+ht)})\, {}_{G}\mathcal{J}_n^{[h]}(v-h) +$ $\sum_{k=0}^{n} \binom{n}{k} \left(\beta_{k,h}\, {}_{G}\mathcal{J}_{n-k}^{[h]}(v) - \frac{1}{2} G_k^{[h]}\, {}_{G}\mathcal{J}_{n-k}^{[h]}(v+1-h) \right)$
IV.	Implicit formulas	${}_{G}\mathcal{J}_n^{[h]}(v) = \sum_{k=0}^{n} \binom{n}{k} (y\|h)_k\, {}_{G}\mathcal{J}_{n-k}^{[h]}(v-y)$ ${}_{G}\mathcal{J}_n^{[h]}(v+y) = \sum_{k=0}^{n} \binom{n}{k} (y\|h)_k\, {}_{G}\mathcal{J}_{n-k}^{[h]}(v)$ ${}_{G}\mathcal{J}_n^{[h]}(v+y) = \sum_{l=0}^{n} \sum_{k=l}^{\infty} \binom{n}{l} S_1(l,k) y^k h^{l-k}\, {}_{G}\mathcal{J}_{n-l}^{[h]}(v)$
V.	Explicit representations	${}_{G}\mathcal{J}_n^{[h]}(v) = \sum_{k=0}^{n} \binom{n}{k} \alpha_{k,h}\, G_{n-k}^{[h]}(v)$ ${}_{G}\mathcal{J}_n^{[h]}(v) = \sum_{k=0}^{n} \binom{n}{k} G_{n-k}^{[h]}\, \mathcal{J}_k(v)$

5. Concluding Remarks

In the following part, we establish the relation of Δ_h FGAP ${}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(v|u)$ with other Δ_h-special polynomials.

Theorem 9. *For $n \geq 1$, the following relation between Δ_h FGAP ${}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(v|u)$ and Δ_h-Genocchi–Appell polynomials ${}_{G}\mathcal{J}_n^{[h]}(v)$ holds true:*

$$ {}_{\mathfrak{F}}\mathcal{J}_n^{[h]}(v|-1) = \frac{1}{n+1}\, {}_{G}\mathcal{J}_{n+1}^{[h]}(v). \tag{63}$$

Proof. Taking $u = -1$ and $r = 1$ in relation (17), we obtain

$$ \sum_{n=0}^{\infty} {}_{\mathfrak{F}}\mathcal{J}_n^{[h]}(v|-1)\frac{t^n}{n!} = \mathcal{J}(t)\frac{1}{t}\left(\frac{2\frac{log(1+ht)}{h}}{(1+ht)^{\frac{1}{h}}+1} \right)(1+ht)^{\frac{v}{h}}. $$

On further simplification, we have

$$ \sum_{n=0}^{\infty} {}_{\mathfrak{F}}\mathcal{J}_n^{[h]}(v|-1)\frac{t^{n+1}}{n!} = \sum_{n=0}^{\infty} {}_{G}A_n^{[h]}(v)\frac{t^n}{n!}. \tag{64}$$

When the coefficients of similar powers of t are compared, assertion (63) follows. □

Similarly, we obtain the following relation between Δ_h FGAP ${}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(v|u)$ and Δ_h-Bernoulli–Appell polynomials ${}_{B}\mathcal{J}_n^{[h]}(v)$:

$$ {}_{\mathfrak{F}}\mathcal{J}_n^{[h]}(v|-1) = \frac{2}{n+1}\, {}_{B}A_{n+1}^{[h]}(v). \tag{65}$$

Remark 4. *Further, on taking $\mathcal{J}(t) = 1$, Δ_h FGAP ${}_{\mathfrak{F}}\mathcal{J}_n^{[r,h]}(v|u)$ could be expressed in the form of the Δ_h-Bernoulli polynomials $B_n^{[h]}(v)$ and Δ_h-Genocchi polynomials $G_n^{[h]}(v)$, respectively:*

$$ {}_{\mathfrak{F}}\mathcal{J}_n^{[h]}(v|-1) = \frac{2}{n+1}\, B_{n+1}^{[h]}(v). \tag{66}$$

$$ {}_{\mathfrak{F}}\mathcal{J}_n^{[h]}(v|-1) = \frac{1}{n+1}\, G_{n+1}^{[h]}(v). \tag{67}$$

Posing a problem. Establish the corresponding results for Δ_h-Bernoulli–Appell polynomials ${}_B\mathcal{J}_n^{[h]}(v)$ and Δ_h-Euler–Appell polynomials ${}_E\mathcal{J}_n^{[h]}(v)$ as in Sections 2 and 3. This posed problem is left to the interested researcher for further investigation.

A significant area of mathematics that has recently drawn the attention of many mathematicians is the study of special functions. Some of the special functions were developed to address particular issues, while others were applied to more general issues. Numerous academics have looked at the Δ_h variants of a few exceptional polynomials. These polynomials are most commonly utilized in the study of finite differences, analytical numerical methods, and applicability in classical calculus and statistics. We refer to [19–21].

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