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ON THE CONTINUITY OF LINEAR CANONICAL BESSEL WAVELET TRANSFORMATIONS

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Abstract. Due to the extra degrees of freedom and simple geometrical manifestation, the linear canonical transform (LCT) has being broadly employed across several disciplines of science and engineering including signal processing, optical and radar systems, electrical and communication systems, quantum physics etc. The main objective of this paper is to study the linear canonical Hankel transformation and the continuous canonical Bessel wavelet transformation and some of their basic properties. The continuous Canonical Bessel wavelet transformation, its inversion formula and Parsevals relation for the continuous Canonical Bessel wavelet transformation are also studied.

1. Introduction

The theory of linear canonical transformation (LCT) was motivated by the work of two different projects by Collins [6] on the field of paraxial optics, on the other hand, Moshinsky and Quesne [11] in the field of nuclear physics in early seventies. The LCT is a four parameter class of linear integral transformation for studying the behaviour of many useful transformations and system responses in physics and engineering in general. Therefore, LCT is found as a powerful mathematical tool in many fields of physics and engineering. In this correspondence, we have defined the continuous Canonical Bessel wavelet transformation and associated properties. A general class of LCT has been studied by[1, 14]. The conventional canonical transformation represents any affine linear transformation in the (x, y) plane and specified by a 2×2 unimodular matrix A(i.e. determinant is one). For the sake of brevity, we may write the matrix as $A = (a, b; c, d)$

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in the text and linear canonical transform of any $f \in L^2(\mathbb{R})$ with respect to the unimodular matrix $A = (a, b, c, d)$ is defined by

$$\mathcal{L}[f](x) = \begin{cases} \int_{\mathbb{R}} f(t) K^A(t,x) dt & b \neq 0 \\ \sqrt{d} \exp \frac{cdx^2}{2} f(dx) & b = 0. \end{cases}$$

where $K^A(t,x)$ is the kernel of linear canonical transform and is given by

$$K^A(t,x) = \frac{1}{\sqrt{2\pi\iota b}} \exp\left\{ \frac{\iota(at^2 - 2tx + dx^2)}{2b} \right\}, \quad b \neq 0$$

The linear canonical transform includes several known transforms as special cases.For example, for $A = (1, b, 0, 1)$, we obtain the Fresnel transform, for $A = (\cos\theta, \sin\theta, -\sin\theta, \cos\theta)$, the LCT boils down to the fractional Fourier transform whereas for $A = (0, 1, -1, 0)$, we obtain the classical Fourier transform. Moreover, Bi-lateral Laplace, Gauss-Weierstrass, and Bargmann transform are also special cases of LCT. Similarly, we can define a linear canonical Hankel transformation, fractional Hankel transformation. For more details we refer to [2, 3, 4, 5, 7, 10, 12, 13] and references therein.

Pathak and Dixit [11] introduced continuous and discrete Bessel wavelet transformations and studied their properties by exploiting the Hankel convolution of Haimo [8] and Hirschman [9] . Upadhyay et al. [15] studied the continuous Bessel wavelet transformation associated with the Hankel-Hausdorff operator.

Let $L^p(\mathbb{R})$ denote the measurable functions f on $\mathbb{R}$ such that the integral $\int_{\mathbb{R}} |f(x)|^p dt$ is a finite. Also let $L^\infty(\mathbb{R})$ be a collection of almost bounded functions endowed with norm

$$||f||_{L^p} = \begin{cases} (\int_{\mathbb{R}} |f(x)|^p)^{1/p}, & 1 \leq p < \infty \\ esssup_{x \in \mathbb{R}} |f(t)|, & p = \infty. \end{cases}$$

The Hankel transformation of H_μ of conventional function $f \in L^1(\mathbb{R}_+), \mathbb{R}_+ = (0, \infty)$ is usually defined by:

$$\hat{f}(y) = (H_\mu f)(y) = \int_0^\infty (xy)^{1/2} J_\mu(xy) f(x) dx \qquad x \in \mathbb{R}_+, \mu \geq -1/2,$$

and its inverse formula is given by

$$f(x) = (H_\mu \hat{f}(y))(x) = \int_0^\infty (xy)^{1/2} J_\mu(xy) \hat{f}(y) dy \qquad y \in \mathbb{R}_+.$$

where J_μ is the Bessel function of the first kind of order μ.

The Linear Canonial Hankel transform (LCHT)is the generalization of conventional Hankel transformation and it is characterize by a unimodal matrix $A = (a, b, c, d)$, where a, b, c and d are four real or complex parameters. The earliest work on canonical Hankel transform was published by Wolf and Bultheel et al. We define a one-dimensional linear canonical Hankel transformation(LCHT)as:

$$\hat{f}^A_\mu(y) = (H^A_\mu f)(y) = \int_0^\infty K^A_\mu(x,y) f(x) dx$$

where the kernel

$$K_\mu^A(x,y) = \frac{e^{-\iota\pi/2(1+\mu)}}{b} e^{\iota/2b(ax^2+dy^2)}(xy)^{1/2}J_\mu(\frac{xy}{b})$$

The inversion formula of one dimensional LCHT is given by

$$f(x) = ((H_\mu^A)^{-1}\hat{f})(x) = \int_0^\infty \overline{K_\mu^A(x,y)}(H_\mu^A f)(y)dy$$

where

$$\overline{K_\mu^A(x,y)} = \frac{e^{\iota\pi/2(1+\mu)}}{b} e^{-\iota/2b(ax^2+dy^2)}(xy)^{1/2}J_\mu(\frac{xy}{b})$$

From [7], wavelets are a family of functions constructed from translation and dilation of a single function ψ are called the mother wavelet defined by

$$\psi_{t,s} = \frac{1}{\sqrt{s}}\psi\left(\frac{x-t}{s}\right), \quad t \in \mathbb{R}, s > 0$$

where s is called the scaling parameter which measures the degree of compression or scale and t is a translation parameter which determines the time location of the wavelet. The linear canonical mother wavelet is defined as

$$\psi_{t,s}^A = \frac{1}{\sqrt{s}}\psi\left(\frac{x-t}{s}\right)e^{\frac{-\iota a}{2b}(x^2-t^2)}$$

for each s, t and A as above. As per [12, 13] the canonical Hankel convolution of $f, \psi \in L^1(\mathbb{R}_+)$ is defined as:

$$(f *_A \psi)(x) = \frac{e^{\frac{-\iota\pi}{2}(1+\mu)}}{b}\int_0^\infty f(x)(\tau_x^A\psi)(y)e^{\frac{\iota a}{2b}y^2}dy$$

where the canonical Hankel translation τ_x^A is given as:

$$\begin{aligned}(\tau_x^A\psi)(y) &= \psi^A(x,y)\\ &= \frac{e^{\frac{-\iota\pi}{2}(1+\mu)}}{b}\int_0^\infty \psi(z)D_\mu^A(x,y,z)e^{\frac{\iota a}{2b}z^2}dz\end{aligned}$$

Where

$$\begin{aligned}D_\mu^A(x,y,z) \quad = \quad & \frac{e^{\frac{-\iota\pi}{2}(1+\mu)}}{b}\int_0^\infty (x\xi)^{\frac{1}{2}}J_\mu\left(\frac{x\xi}{b}\right)(y\xi)^{\frac{1}{2}}J_\mu\left(\frac{y\xi}{b}\right)(z\xi)^{\frac{1}{2}}\\ & \times J_\mu\left(\frac{z\xi}{b}\right)e^{\frac{-\iota a}{2b}(x^2+y^2+z^2)}\xi^{1+2\mu}d\xi.\end{aligned}$$

and

$$\frac{e^{\frac{-\iota\pi}{2}(1+\mu)}}{b}\int_0^\infty D_\mu^A(x,y,z)e^{\frac{\iota a}{2b}z^2}z^{1+2\mu}dz = \frac{e^{\frac{-\iota a}{2b}(x^2+y^2)}}{(2b)^\mu\Gamma(\mu+1)}. \tag{1.1}$$

Lemma 1.1. *If $f \in L^2(\mathbb{R}+)$, then*

$$\left|\left|x^{-\mu-1/2}(\tau_x^A f)(y)\right|\right|_{L^2} = \frac{1}{|b^{\mu+1/2}2^\mu\Gamma(\mu+1)|}||f||_{L^2}$$

Proof. Since

$$\begin{aligned}(\tau_x^A f)(y) &= f^A(x,y)\\ &= \frac{e^{\frac{-\iota\pi}{2}(1+\mu)}}{b}\int_0^\infty f(z)D_\mu^A(x,y,z)e^{\frac{\iota a}{2b}z^2}dz\end{aligned}$$

Using (1.1), we have

$$\begin{aligned}&\left|(\tau_x^A f)(y)\right|\\ &\le \int_0^\infty \left|f(z)z^{-1/2(\mu+1/2)}\left\{D_\mu^A(x,y,z)\right\}^{1/2} z^{1/2(\mu+1/2)}\left\{D_\mu^A(x,y,z)\right\}^{1/2}\right|dz\\ &\le \left(\int_0^\infty z^{-(\mu+1/2)}|f(z)|^2|D_\mu^A(x,y,z)|dz\right)^{1/2}\\ &\quad\left(\int_0^\infty z^{(\mu+1/2)}|f(z)|^2|D_\mu^A(x,y,z)|dz\right)^{1/2}\\ &\le \left(\frac{(xy)^{\mu+1/2}}{|(b)^{\mu+1/2}|2^\mu\Gamma(\mu+1)}\right)^{1/2}\\ &\quad\left(\int_0^\infty z^{-(\mu+1/2)}|f(z)|^2|D_\mu^A(x,y,z)|dz\right)^{1/2} \qquad (1.2)\end{aligned}$$

so that

$$\begin{aligned}&\int_0^\infty \left|(\tau_x^A f)(y)\right|^2 dy\\ &\le \frac{x^{\mu+1/2}}{|(b)^{\mu+1/2}|2^\mu\Gamma(\mu+1)}\int_0^\infty z^{-(\mu+1/2)}|f(z)|^2dz\int_0^\infty |D_\mu^A(x,y,z)|y^{\mu+1/2}dy\\ &\le \frac{x^{\mu+1/2}}{|(b)^{\mu+1/2}|2^\mu\Gamma(\mu+1)}\int_0^\infty z^{-(\mu+1/2)}|f(z)|^2\frac{zx^{\mu+1/2}}{|(b)^{\mu+1/2}|2^\mu\Gamma(\mu+1)}dz\\ &= \frac{x^{2\mu+1/2}}{(|(b)^{\mu+1/2}|2^\mu\Gamma(\mu+1))^2}\int_0^\infty |f(z)|^2dz\end{aligned}$$

therefore

$$\left|\left|x^{-\mu-1/2}(\tau_x^A f)(y)\right|\right|_{L^2} = \frac{1}{|b^{\mu+1/2}2^\mu\Gamma(\mu+1)|}||f||_{L^2}.$$

□

Remark 1.2. If $f \in L^2(\mathbb{R}+)$, then

$$\int_0^\infty \left|(\tau_y^A f)(x)\right|^2 dx \le \frac{y^{2\mu+1/2}}{(|(b)^{\mu+1/2}|2^\mu\Gamma(\mu+1))^2}\int_0^\infty |f(z)|^2dz,$$

and

$$\left|\left|y^{-\mu-1/2}(\tau_y^A f)(x)\right|\right|_{L^2} = \frac{1}{|b^{\mu+1/2}2^\mu\Gamma(\mu+1)|}||f||_{L^2}.$$

Proposition 1.3. Let $(H_\mu^A f)(y)$ denote the canonical Hankel transformations of function f then

$$\int_0^\infty f(x)\overline{g(x)}dx = b\int_0^\infty (H_\mu^A f)(y)\overline{(H_\mu^A g)(y)}dy \tag{1.3}$$

and

$$\int_0^\infty |f(x)|^2\,dx = b\int_0^\infty \left|(H_\mu^A f)(y)\right|^2 dy. \tag{1.4}$$

Proof. We have

$$\begin{aligned}
&\langle f, g\rangle \\
&= \int_0^\infty f(x)\overline{g(x)}d(x) \\
&= \int_0^\infty f(x)\left(\overline{\left(\frac{e^{\frac{\iota\pi}{2}(1+\mu)}}{b}\right)} b\overline{\int_0^\infty e^{\frac{-\iota}{2b}(ax^2+dy^2)}\left(\frac{xy}{b}\right)^{1/2} J_\mu\left(\frac{xy}{b}\right)(H_\mu^A g)(y)dy}\right) \\
&= b\int_0^\infty \left(\frac{e^{\frac{-\iota\pi}{2}(1+\mu)}}{b}\right)\overline{(H_\mu^A g)(y)}\left(\int_0^\infty e^{\frac{\iota}{2b}(ax^2+dy^2)}\left(\frac{xy}{b}\right)^{1/2} J_\mu\left(\frac{xy}{b}\right)f(x)dx\right)dy \\
&= b\int_0^\infty (H_\mu^A f)(y)\overline{(H_\mu^A g)(y)}dy
\end{aligned}$$

if $f = g$, then

$$\int_0^\infty |f(x)|^2\,dx = b\int_0^\infty \left|(H_\mu^A f)(y)\right|^2 dy.$$

□

2. The continuous Linear Canonical Bessel wavelet transformation(CLCBWT)

The continuous linear canonical Bessel wavelet transformation (CLCBWT) is a generalization of the ordinary continuous Bessel wavelet transformation(CBWT). In this section, we define the continuous linear canonical Bessel wavelet transformation and study some of its properties using the theory of linear canonical Hankel convolution.

A Canonical Bessel wavelet is a function $\psi \in L^2(\mathbb{R})$ which satisfies the condition

$$\mathcal{C}_{\mu,\psi}^A = \int_0^\infty \frac{|(H_\mu^A\psi)(x)|^2}{x^{2(\mu-1)}}dx < \infty, \quad \mu \geq -1/2$$

where $\mathcal{C}_{\mu,\psi}^A$is called the admissibility condition of the canonical Bessel wavelet and $(H_\mu^A\psi)$is the canonical Bessel transformation of ψ. The canonical Bessel wavelets ψ_t^A, s

are generated from one single function $\psi \in L^2(\mathbb{R}_+)$ by dilation and translation with parameters $s > 0$ and $t > 0$ respectively by

$$\psi^A_{t,s}(x) = s^{-1/2} D_s \tau^A_t \psi(x) = s^{-1/2} D_s \psi^A(t,x) = s^{-1/2} \psi^A\left(\frac{t}{s}, \frac{x}{s}\right)$$
$$= s^{-1/2} e^{\frac{-\iota a}{2b}(\frac{t^2}{s^2}+\frac{x^2}{s^2})} \int_0^\infty \psi(z) D^A_\mu \left(\frac{t}{s}, \frac{x}{s}, z\right) dz.$$

Lemma 2.1. *If* $\psi \in L^2(\mathbb{R}+)$, *then*

$$\left\|\psi^A_{t,s}\right\|_{L^2} = \frac{t^{(\mu+1/2)} s^{-(\mu+1/2)}}{|b^{\mu+1/2} 2^\mu \Gamma(\mu+1)|} ||\psi||_{L^2}$$

Proof. Since

$$\psi^A_{t,s} = s^{-1/2} e^{\frac{-\iota a}{2b}(\frac{t^2}{s^2}+\frac{x^2}{s^2})} \int_0^\infty \psi(z) D^A_\mu \left(\frac{t}{s} + \frac{x}{s}, z\right) dz$$

Now,

$$\left|\psi^A_{t,s}\right| \le s^{-1/2} \int_0^\infty |\psi(z)| z^{-1/2(\mu+1/2)} \left|\left\{D^A_\mu\left(\frac{t}{s}, \frac{x}{s}, z\right)\right\}^{1/2}\right| z^{1/2(\mu+1/2)}$$
$$\times \left|\left\{D^A_\mu\left(\frac{t}{s}, \frac{x}{s}, z\right)\right\}^{1/2}\right| dz$$
$$\le s^{-1/2} \left(\int_0^\infty |\psi(z)|^2 z^{-(\mu+1/2)} \left|D^A_\mu\left(\frac{t}{s}, \frac{x}{s}, z\right)\right| dz\right)^{1/2}$$
$$\times \left(\int_0^\infty \left|D^A_\mu\left(\frac{t}{s}, \frac{x}{s}, z\right)\right| z^{(\mu+1/2)} dz\right)^{1/2}$$
$$\le s^{-1/2} \left(\frac{(tx)^{\mu+1/2}}{s^{2(\mu+1/2)}|b^{\mu+1/2}|2^\mu\Gamma(\mu+1)}\right)^{1/2}$$
$$\times \left(\int_0^\infty |\psi(z)|^2 z^{-(\mu+1/2)} \left|D^A_\mu\left(\frac{t}{s}, \frac{x}{s}, z\right)\right| dz\right)^{1/2}.$$

Thus

$$\int_0^\infty |\psi^A_{t,s}(x)|^2 dx$$
$$\le \frac{t^{2(\mu+1/2)}}{s^{2\mu+2}|b^{\mu+1/2}|2^\mu\Gamma(\mu+1)} \int_0^\infty z^{-(\mu+1/2)} |\psi(z)|^2 dz \int_0^\infty \left|D^A_\mu\left(\frac{t}{s}, \frac{x}{s}, z\right)\right| x^{\mu+1/2} dx$$
$$\le \frac{t^{2(\mu+1/2)}}{(s^{\mu+1/2}|b^{\mu+1/2}|2^\mu\Gamma(\mu+1))^2} \int_0^\infty |\psi(z)|^2 dz.$$

Hence

$$||\psi^A_{t,s}||_{L^2} \le \frac{t^{(\mu+1/2)} s^{-(\mu+1/2)}}{(|b^{\mu+1/2}|2^\mu\Gamma(\mu+1))^2} ||\psi||_{L^2}.$$

□

Theorem 2.2. *Let $f, \psi \in L^2(\mathbb{R}+)$. Then the continuous linear canonical Bessl wavelet transform B^A_ψ is defined on f by*

$$\left(B^A_\psi f\right)(t,s) = e^{-\iota\pi/2(1+\mu)} s^{-\mu} \int_0^\infty e^{\frac{-\iota a}{2b}(\frac{1}{s^2}-1)(sx)^2} x^{-\mu-1/2} \left(\frac{tx}{b}\right)^{1/2} \times J_\mu\left(\frac{tx}{b}\right) (H^A_\mu e^{\frac{\iota a}{2b}(.)} f)(x) \overline{(H^A_\mu \psi)}(sx) dx.$$

Proof. We have

$$\begin{aligned}
&\left(B^A_\psi f\right)(t,s) \\
&= \langle f, \psi^A_{t,s} \rangle \\
&= \int_0^\infty f(\xi) \overline{\psi^A_{t,s}(\xi)} d\xi \\
&= \int_0^\infty f(\xi) \overline{\left(s^{-1/2} e^{\frac{-\iota a}{2b}(\frac{t^2}{s^2}+\frac{\xi^2}{s^2})} \int_0^\infty \psi(z) D^A_\mu\left(\frac{t}{s}, \frac{\xi}{s}, z\right) dz\right)} d\xi \\
&= \frac{b}{e^{\iota\pi/2(1+\mu)} s^{1/2}} \int_0^\infty \left(\frac{e^{\iota\pi/2(1+\mu)}}{b} \int_0^\infty e^{\frac{\iota a}{2b}(\xi^2+\frac{\omega^2}{s^2}} \left(\frac{\xi\omega}{sb}\right)^{1/2} J_\mu\left(\frac{\xi\omega}{sb}\right) e^{\frac{-\iota a \xi^2}{2b}} f(\xi) d\xi\right) \\
&\quad \times e^{\frac{-\iota a\omega^2}{b}(\frac{1}{s^2}-1)} \omega^{(-\mu-1/2)} \left(\frac{t\omega}{sb}\right)^{1/2} J_\mu\left(\frac{t\omega}{sb}\right) \overline{(H^A_\mu \psi(z))}(\omega) d\omega \\
&= \frac{b}{e^{\iota\pi/2(1+\mu)} s^{1/2}} \int_0^\infty e^{\frac{-\iota a\omega^2}{b}(\frac{1}{s^2}-1)} \omega^{(-\mu-1/2)} \left(\frac{t\omega}{sb}\right)^{1/2} \\
&\quad \times J_\mu\left(\frac{t\omega}{sb}\right) (H^A_\mu e^{\frac{-\iota a}{2b}(.)^2} f)\left(\frac{\omega}{s}\right) \overline{(H^A_\mu \psi)}(\omega) d\omega.
\end{aligned}$$

By taking $\frac{\omega}{s} = x$ the continuous linear canonical Bessel wavelet transformation can be written as

$$\left(B^A_\psi f\right)(t,s) = e^{-\iota\pi/2(1+\mu)} s^{-\mu} \int_0^\infty e^{\frac{-\iota a}{2b}(\frac{1}{s^2}-1)(sx)^2} x^{-\mu-1/2} \left(\frac{tx}{b}\right)^{1/2} \times J_\mu\left(\frac{tx}{b}\right) (H^A_\mu e^{\frac{\iota a}{2b}(.)^2} f)(x) \overline{(H^A_\mu \psi)}(sx) dx.$$

This means

$$H^A_\mu \left\{ e^{\frac{-\iota a}{2b} t^2} (B^A_\psi f)(t,s) \right\} = b s^{-\mu} (x^{-\mu-1/2} e^{\iota \frac{s^2}{2b} a x^2} (H^A_\mu e^{-\iota \frac{a}{2b}(.)^2} f)(x) \overline{(H^A_\mu \psi)}(sx)).$$

□

Theorem 2.3. *If ψ_1 and ψ_2 are two wavelets and $(B^A_{\psi_1} f)(t,s)$ and $(B^A_{\psi_2} g)(t,s)$ denote the continuous linear canonical Bessel wavelet transformations of $f, g \in L^2(\mathbb{R}_+)$ respectively, then*

$$\int_0^\infty (B^A_{\psi_1} f)(t,s) \overline{(B^A_{\psi_2} g)}(t,s) \frac{dt ds}{s^2} = b^2 \mathcal{C}^A_{\mu,\psi_1,\psi_2} \langle f, g \rangle,$$

where

$$\mathcal{C}^A_{\mu,\psi_1,\psi_2} = \int_0^\infty \frac{\overline{(H^A_\mu\psi_1)}(s)(H^A_\mu\psi_2)(s)}{s^{2(\mu+1)}}ds < \infty.$$

Proof. Since

$$\left(B^A_\psi f\right)(t,s) = e^{-\iota\pi/2(1+\mu)}s^{-\mu}\int_0^\infty e^{\frac{-\iota a}{2b}(\frac{1}{s^2}-1)(sx)^2}x^{-\mu-1/2}\left(\frac{tx}{b}\right)^{1/2} \times J_\mu\left(\frac{tx}{b}\right)(H^A_\mu e^{\frac{\iota a}{2b}(.)^2}f)(x)\overline{(H^A_\mu\psi_1)}(sx)dx.$$

Now

$$\begin{aligned}
&\int_0^\infty (B^A_{\psi_1}f)(t,s)\overline{(B^A_{\psi_2}g)}(t,s)\frac{dtds}{s^2}\\
&= \int_0^\infty\int_0^\infty \Bigg[e^{-\iota\pi/2(1+\mu)}s^{-\mu}\int_0^\infty e^{\frac{-\iota a}{2b}(\frac{1}{s^2}-1)(sx)^2}x^{-\mu-1/2}\left(\frac{tx}{b}\right)^{1/2}\\
&\quad\times J_\mu\left(\tfrac{tx}{b}\right)(H^A_\mu e^{\frac{\iota a}{2b}(.)}f)(x)\overline{(H^A_\mu\psi_1)}(sx)dx\Bigg]\\
&\quad\times\overline{\Big[e^{-\iota\pi/2(1+\mu)}s^{-\mu}\textstyle\int_0^\infty e^{\frac{-\iota a}{2b}(\frac{1}{s^2}-1)(sx)^2}x^{-\mu-1/2}\left(\frac{tx}{b}\right)^{1/2}}\\
&\quad\overline{\times J_\mu\left(\tfrac{ty}{b}\right)(H^A_\mu e^{\frac{\iota a}{2b}(.)^2}f)(y)\overline{(H^A_\mu\psi_2)}(sy)dy\Big]}dt\tfrac{ds}{s^2}\\
&= \int_0^\infty\int_0^\infty b^3 s^{-2(\mu+1)}e^{\frac{-\iota a}{2b}(2-s^2)x^2}x^{-\mu-1/2}(H^A_\mu e^{\frac{-\iota a}{2b}(.)^2}f)(x)\overline{(H^A_\mu\psi_1)}(sx)\\
&\quad\times\Big\{\tfrac{e^{\iota\pi/2(1+\mu)}}{b}\textstyle\int_0^\infty e^{\frac{\iota a}{2b}(t^2+x^2)}\left(\frac{tx}{b}\right)^{1/2}J_\mu\left(\frac{tx}{b}\right)\\
&\quad\times\Big(\tfrac{e^{-\iota\pi/2(1+\mu)}}{b}\textstyle\int_0^\infty e^{\frac{-\iota a}{2b}(t^2+y^2)}\left(\frac{tx}{b}\right)^{1/2}J_\mu\left(\frac{ty}{b}\right)\\
&\quad\times e^{\frac{-\iota a}{2b}(2-s^2)y^2}y^{-\mu-1/2}\overline{(H^A_\mu e^{\frac{-\iota a}{2b}(.)^2}g)(y)}(H^A_\mu\psi_2)(sy)dy\Big)dt\Big\}dxds\\
&= \int_0^\infty\int_0^\infty b^3 s^{-2(\mu+1)}e^{\frac{-\iota a}{2b}(2-s^2)x^2}x^{-\mu-1/2}(H^A_\mu e^{\frac{-\iota a}{2b}(.)^2}f)(x)\overline{(H^A_\mu\psi_1)}(sx)\\
&\quad\times\Big\{\tfrac{e^{\iota\pi/2(1+\mu)}}{b}\textstyle\int_0^\infty e^{\frac{\iota a}{2b}(t^2+x^2)}\left(\frac{tx}{b}\right)^{1/2}J_\mu\left(\frac{tx}{b}\right)(H^A_\mu)^{-1}\Big(e^{\frac{-\iota a}{2b}(2-s^2)y^2}\\
&\quad\times y^{-\mu-1/2}\overline{(H^A_\mu e^{\frac{-\iota a}{2b}(.)^2}g)}(y)(H^A_\mu\psi_2)(sy)\Big)(t)dt.\Big\}dxds.
\end{aligned}$$

$$= b^3 \int_0^\infty \int_0^\infty s^{-2(\mu+1)} e^{\frac{-\iota a}{2b}(2-s^2)x^2} x^{-\mu-1/2} (H_\mu^A e^{\frac{-\iota a}{2b}(.)^2} f)(x) \overline{(H_\mu^A \psi_1)}(sx)$$

$$\times (H_\mu^A)(H_\mu^A)^{-1} \left(e^{\frac{-\iota a}{2b}(2-s^2)y^2} y^{-\mu-1/2} \overline{(H_\mu^A e^{\frac{-\iota a}{2b}(.)^2} g)}(y)(H_\mu^A \psi_2)(sy) \right)(x) dx ds$$

$$= b^3 \int_0^\infty \int_0^\infty s^{-2(\mu+1)} x^{-2\mu-1} (H_\mu^A e^{\frac{-\iota a}{2b}(.)^2} f)(x)$$

$$\times \overline{(H_\mu^A e^{\frac{-\iota a}{2b}(.)^2} g)}(x)(H_\mu^A \psi_2)(sx) dx ds$$

$$= b^3 \int_0^\infty (H_\mu^A e^{\frac{-\iota a}{2b}(.)^2} f)(x) \overline{(H_\mu^A e^{\frac{-\iota a}{2b}(.)^2} g)}(x)$$

$$\times \left(\int_0^\infty (sx)^{-2(\mu+1)} \overline{(H_\mu^A \psi_1)}(sx)(H_\mu^A \psi_2)(sx) x ds \right) dx$$

$$= b^3 C^A_{\mu,\psi_1,\psi_2} \int_0^\infty (H_\mu^A f)(x) \overline{(H_\mu^A g)}(x) dx$$

$$= b^3 C^A_{\mu,\psi_1,\psi_2} \langle H_\mu^A f, H_\mu^A g \rangle$$

$$= b^2 \mathcal{C}^A_{\mu,\psi_1,\psi_2} \langle f, g \rangle$$

Hence the proof is completed. □

Theorem 2.4. *If ψ is a wavelet and $(B_\psi^A f)(t,s)$ and $(B_\psi^A f)(t,s)$ are the continuous linear canonical Bessel wavelet transform of $f, g \in L^2(\mathbb{R}_+)$ respectively,Then*

$$\int_0^\infty \int_0^\infty (B_\psi^A f)(t,s) \overline{(B_\psi^A g)}(t,s) \frac{dtds}{s^2} = b^2 \mathcal{C}^A_{\mu,\psi} \langle f, g \rangle.$$

Proof. If we take $\psi_1 = \psi_2 = \psi$ in Theorem 2.3 we get the proof of Theorem 2.6 □

Remark 2.5. If $f = g$ and $\psi_1 = \psi_2 = \psi$ then from Theorem 2.6, we have

$$\int_0^\infty \int_0^\infty |(B_\psi^A f)(t,s)|^2 \frac{dtds}{s^2} = b^2 \mathcal{C}^A_{\mu,\psi} ||f||_2^2$$

Theorem 2.6. *Let $f \in L^2(\mathbb{R}_+)$.Then f can be reconstructed by the formula*

$$f(\xi) = \frac{1}{b^2 \mathcal{C}^A_{\mu\psi}} \int_0^\infty \int_0^\infty (B_\psi^A f)(t,s) \psi^A_{t,s}(\xi) \frac{dtds}{s^2}, \quad s > 0$$

Proof. For any $g \in L^2(\mathbb{R}_+)$, we have

$$\begin{aligned} b^2\mathcal{C}^A_{\mu,\psi}\langle f, g\rangle &= \int_0^\infty \int_0^\infty (B^A_\psi f)(t,s)\overline{(B^A_\psi g)}(t,s)\frac{dtds}{s^2} \\ &= \int_0^\infty \int_0^\infty (B^A_\psi f)(t,s)\overline{\left(\int_0^\infty g(\xi)\overline{\psi^A_{t,s}(\xi)}d\xi\right)}\frac{dtds}{s^2} \\ &= \int_0^\infty \left[\int_0^\infty \int_0^\infty (B^A_\psi f)(t,s)\psi^A_{t,s}(\xi)\frac{dtds}{s^2}\right]\overline{g(\xi)}d\xi \\ &= \left\langle \int_0^\infty \int_0^\infty (B^A_\psi f)(t,s)\psi^A_{t,s}(\xi)\frac{dtds}{s^2}, g(\xi)\right\rangle. \end{aligned}$$

Therefore,

$$f(\xi) = \frac{1}{b^2\mathcal{C}^A_{\mu\psi}}\int_0^\infty \int_0^\infty (B^A_\psi f)(t,s)\psi^A_{t,s}(\xi)\frac{dtds}{s^2}.$$

□

Theorem 2.7. *If $\psi \in L^2(\mathbb{R}_+)$, then*

$$\int_0^\infty \left[(B^A_\psi f)(t,s)\overline{(B^A_\psi g)}(t,s)\right]dt = b^2 s^{-2\mu}\langle \mathcal{P}, Q\rangle,$$

where

$$\mathcal{P}(x) = e^{\frac{-\iota a}{2b}(2-s^2)x^2}x^{-\mu-1/2}(H^A_\mu e^{\frac{-\iota a}{2b}(.)^2}f)(x)\overline{(H^A_\mu\psi)}(sx),$$

$$\mathcal{Q}(x) = e^{\frac{-\iota a}{2b}(2-s^2)y^2}y^{-\mu-1/2}(H^A_\mu e^{\frac{-\iota a}{2b}(.)^2}\overline{g})(y)\overline{(H^A_\mu\psi)}(sy).$$

Proof. Using Theorem 2.2 and Theorem 2.7, we have

$$\begin{aligned} &\int_0^\infty \left[(B^A_\psi f)(t,s)\overline{(B^A_\psi g)}(t,s)\right]dt \\ &= \int_0^\infty \langle f, \psi^A_{t,s}\rangle\,\overline{\langle g, \psi^A_{t,s}\rangle}dt \\ &= \int_0^\infty \left(\int_0^\infty f(\xi)\overline{\psi^A_{t,s}}(\xi)d\xi\right)\left(\int_0^\infty \overline{g}(\eta)\psi^A_{t,s}(\eta)d\eta\right)dt \\ &= \int_0^\infty \left[\int_0^\infty f(\xi)\left(\overline{s^{-1/2}e^{\frac{-\iota a}{2b}(\frac{t^2}{s^2}+\frac{\xi^2}{s^2})}\int_0^\infty \psi(z)D^A_\mu\left(\frac{t}{s},\frac{\xi}{s},z\right)dz}\right)d\xi\right] \\ &\quad \times \left[\int_0^\infty \overline{g}(\eta)\left(s^{-1/2}e^{\frac{-\iota a}{2b}(\frac{t^2}{s^2}+\frac{\eta^2}{s^2})}\int_0^\infty \psi(z)D^A_\mu\left(\frac{t}{s},\frac{\eta}{s},z\right)dz\right)d\eta\right]dt \end{aligned}$$

$$= b^3 s^{-2\mu} \int_0^\infty H_\mu^A (e^{\frac{-\iota a}{2b}(2-s^2)x^2} x^{-\mu-1/2} (H_\mu^A e^{\frac{-\iota a}{2b}(.)^2} f)(x) \overline{(H_\mu^A \psi)}(sx))(t)$$
$$\times \overline{H_\mu^A (e^{\frac{-\iota a}{2b}(2-s^2)y^2} y^{-\mu-1/2} (H_\mu^A e^{\frac{-\iota a}{2b}(.)^2} \overline{g})(y) \overline{(H_\mu^A \psi)}(sy))}(t) dt$$
$$= b^3 s^{-2\mu} \int_0^\infty (H_\mu^A \mathcal{P})(t) \overline{(H_\mu^A \mathcal{Q})}(t) dt$$
$$= b^3 s^{-2\mu} \left\langle H_\mu^A \mathcal{P}, H_\mu^A \mathcal{Q} \right\rangle$$
$$= b^3 s^{-2\mu} \left\langle \mathcal{P}, \mathcal{Q} \right\rangle .$$

This completes the proof of the theorem. □

Theorem 2.8. *If ψ be in $L^2(\mathbb{R}_+)$ is a Bessel wavelet and f is a function which is bounded and integrable ,then the convolution $(\psi *_A f)(x)$ is linear canonical Bessel wavelet.*

Proof. We know that

$$(\psi *_A f)(x) = \int_0^\infty (\tau_x^A \psi)(y) y^{-(\mu+1/2)} f(y) dy.$$

Therefore,

$$|(\psi *_A f)(x)| \leq \textstyle\int_0^\infty \left|(\tau_x^A \psi)(y) y^{-(\mu+1/2)} f^{1/2}(y)\right| \left|f^{1/2}\right| dy,$$

$$\Rightarrow \int_0^\infty |(\psi *_A f)(x)|^2 dx$$
$$\leq \int_0^\infty \left(\int_0^\infty \left| y^{-(\mu+1/2)} f^{1/2}(y) (\tau_x^A \psi)(y) \right| |f^{1/2}| dy \right)^2 dx$$
$$\leq \int_0^\infty \left[\left(\int_0^\infty |f(y)| |y^{-(\mu+1/2)} (\tau_x^A \psi)(y)|^2 dy \right)^{1/2} \left(\int_0^\infty |f(y)| dy \right)^{1/2} \right]^2 dx$$
$$= \left(\int_0^\infty |f(y)| dy \right) \left(\int_0^\infty \left(\int_0^\infty |f(y)| |y^{-(\mu+1/2)} (\tau_x^A \psi)(y)|^2 dx \right) |f(y) dy| \right)$$
$$\leq \left(\int_0^\infty |f(y) dy| \right)^2 \left(\int_0^\infty |y^{-(\mu+1/2)} (\tau_y^A \psi)(x)|^2 dx \right)$$

This implies that

$$\|(\psi *_A f)(x)\|_{L^2} \leq \frac{1}{|(b)^{\mu+1/2}| 2^\mu \Gamma(\mu+1)} \|\psi\|_{L^2} \|f\|_{L^1} < \infty.$$

We have $(\psi *_A f) \in L^2(\mathbb{R}_+)$. Moreover,

$$\begin{aligned}&\int_0^\infty x^{-2(\mu+1)} \left|H_\mu^A(\psi *_A f)(x)\right|^2 dx\\ &\le \frac{1}{|b|}\int_0^\infty \left|x^{-3\mu-5/2}\left(H_\mu^A f\right)\left(\frac{x}{b}\right)\left(H_\mu^A \psi\right)\left(\frac{x}{b}\right)\right|^2 dx\\ &\le \frac{1}{|b|} sup\left(x^{-\mu-1/2}\left|\left(H_\mu^A f\right)\left(\frac{x}{b}\right)\right|^2\right)\\ &\qquad \times \int_0^\infty x^{-2(\mu+1)}\left|\left(H_\mu^A \psi\right)\left(\frac{x}{b}\right)\right|^2 dx.\\ &= \mathcal{C}_{\mu,\psi}^A sup\left(x^{-\mu-1/2}\left|\left(H_\mu^A f\right)\left(\frac{x}{b}\right)\right|^2\right) < \infty.\end{aligned}$$

Thus, the convolution function$(\psi *_A f)$ is a linear canonical Bessel wavelet. □

Theorem 2.9. *Let f, ψ are in $L^2(\mathbb{R}_+)$ and $(B_\psi^A f)(t,s)$is a continuous linear canonical Bessel wavelet transform(CLCBWT), then*

(1) $(B_\psi^A f)(t,s)$ is continuous on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$.

(2) $\|(B_\psi^A f)(t,s)\|_{L^\infty} \le \dfrac{t^{\mu+1/2} s^{-(\mu+1/2)}}{|b^{\mu+1/2}|2^\mu \Gamma(\mu+1)}\|f\|_L^2 \|\psi\|_{L^2}$

Proof. (1) Let (t_0, s_0) be an arbitrary but fixed in $\mathbb{R}_+$.Then using Hölders inequality we have:

$$\begin{aligned}&\left|(B_\psi^A f)(t,s) - (B_\psi^A f)(t_0,s_0)\right|\\ &\le s^{-1/2}\int_0^\infty\int_0^\infty \left|f(x)\psi(z)\left[D_\mu^A\left(\frac{t}{s},\frac{x}{s},z\right)\right.\right.\\ &\qquad \left.\left.-D_\mu^A\left(\frac{t_0}{s_0},\frac{x}{s_0},z\right)\right]\right| dxdz\\ &= s^{-1/2}\left(\int_0^\infty x^{-\mu-1/2}|f(x)|^2 dx \int_0^\infty z^{\mu+1/2}\right.\\ &\qquad \left.\times\left[\left|D_\mu^A\left(\frac{t}{s},\frac{x}{s},z\right) - D_\mu^A\left(\frac{t_0}{s_0},\frac{x}{s_0},z\right)\right|\right] dz\right)^{1/2}\\ &\qquad \times \left(\int_0^\infty z^{-\mu-1/2}|f(\psi)|^2 dz\int_0^\infty x^{\mu+1/2}\right.\\ &\qquad \left.\times\left[\left|D_\mu^A\left(\frac{t}{s},\frac{x}{s},z\right) - D_\mu^A\left(\frac{t_0}{s_0},\frac{x}{s_0},z\right)\right|\right] dx\right)^{1/2}\end{aligned}$$

As

$$\begin{aligned}&\int_0^\infty z^{\mu+1/2}\left[\left|D_\mu^A\left(\frac{t}{s},\frac{x}{s},z\right) - D_\mu^A\left(\frac{t_0}{s_0},\frac{x}{s_0},z\right)\right|\right] dz\\ &\qquad\qquad \le \frac{x^{\mu+1/2}}{|b^{\mu+1/2}|2^\mu\Gamma(\mu+1)}\left[\frac{t^{(\mu+1/2)}}{s^{2(\mu+1/2)}} - \frac{t_0^{(\mu+1/2)}}{s_0^{2(\mu+1/2)}}\right],\end{aligned}$$

And

$$\int_0^\infty x^{\mu+1/2}\left[\left|D_\mu^A\left(\tfrac{t}{s},\tfrac{x}{s},z\right)-D_\mu^A\left(\tfrac{t_0}{s_0},\tfrac{x}{s_0},z\right)\right|\right]dz$$

$$\leq \frac{z^{\mu+1/2}}{|b^{\mu+1/2}|2^\mu\Gamma(\mu+1)}\left[st^{(\mu+1/2)}-s_0t_0^{(\mu+1/2)}\right],$$

Hence by dominated convergence theorem and by the continuity of $D_\mu^A\left(\frac{t}{s},\frac{x}{s},z\right)$ in the variables t and s ,we have

$$\lim_{t\to t_0}\lim_{s\to s_0}\left|(B_\psi^A f)(t,s)-(B_\psi^A f)(t_0,s_0)\right|=0$$

Which proves $(B_\psi^A f)(t,s)$ is continuous on $\mathbb{R}_+\times\mathbb{R}_+$.

□

Proof. (2)We have

$$(B_\psi^A f)(t,s)=\int_0^\infty f(x)\overline{\left(s^{-1/2}e^{\frac{-\iota a}{2b}\left(\frac{t^2}{s^2}+\frac{x^2}{s^2}\right)}\int_0^\infty \psi(z)D_\mu^A\left(\frac{t}{s},\frac{x}{s},z\right)dz\right)}dx$$

Therefore, by the Hölders inequality, we have

$$\begin{aligned}
&|(B_\psi^A f)(t,s)|\\
&\leq s^{-1/2}\left(\int_0^\infty\int_0^\infty x^{-(\mu+1/2)}|f(x)|^2 z^{(\mu+1/2)}\left|D_\mu^A\left(\frac{t}{s},\frac{x}{s},z\right)\right|dxdz\right)^{1/2}\\
&\quad\times\left(\int_0^\infty\int_0^\infty z^{-(\mu+1/2)}|\psi(z)|^2 x^{(\mu+1/2)}\left|D_\mu^A\left(\frac{t}{s},\frac{x}{s},z\right)\right|dxdz\right)^{1/2}\\
&= s^{-1/2}\left(\int_0^\infty x^{-(\mu+1/2)}|f(x)|^2dx\int_0^\infty\left|D_\mu^A\left(\frac{t}{s},\frac{x}{s},z\right)\right|z^{(\mu+1/2)}dz\right)^{1/2}\\
&\quad\times\left(\int_0^\infty z^{-(\mu+1/2)}|\psi(z)|^2dz\int_0^\infty\left|D_\mu^A\left(\frac{t}{s},\frac{x}{s},z\right)\right|x^{(\mu+1/2)}dx\right)^{1/2}\\
&\quad\leq s^{-1/2}\left(\frac{t^{\mu+1/2}}{s^{2(\mu+1/2)}|b^{\mu+1/2}|2^\mu\Gamma(\mu+1)}\right)^{1/2}\left(\int_0^\infty|f(x)|^2dx\right)^{1/2}
\end{aligned}$$

$$\times \left(\frac{st^{\mu+1/2}}{|b^{\mu+1/2}|2^{\mu}\Gamma(\mu+1)}\right)^{1/2}\left(\int_0^{\infty}|\psi(z)|^2dz\right)^{1/2}$$

$$= \frac{t^{\mu+1/2}}{s^{(\mu+1/2)}|b^{\mu+1/2}|2^{\mu}\Gamma(\mu+1)}\|f\|_{L^2}\|\psi\|_{L^2}$$

$$= \frac{t^{\mu+1/2}s^{-(\mu+1/2)}}{s^{(\mu+1/2)}|b^{\mu+1/2}|2^{\mu}\Gamma(\mu+1)}\|f\|_{L^2}\|\psi\|_{L^2}.$$

This completes the proof. □

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References

[1] L. M. Bernard, ABCD matrix formalism of fractional Fourier optics, Opt. Eng. 35 (1996), no. 3, 732–740.

[2] M. Y. Bhat and A. H. Dar, Vector-valued nonuniform multiresolution analysis associated with linear canonical transform, Filomat 37 (2023), no. 16, 5165–5180.

[3] M. Y. Bhat and A. H. Dar, Scaled Wigner distribution in the offset linear canonical domain, Optik, 262 (2022), Art. ID 169286, 1–11.

[4] M. Y. Bhat and A. H. Dar, Fractional vector-valued nonuniform MRA and associated wavelet packets on $L^2(\mathbb{R},\mathbb{C}^M)$, Frac. Calc. Appl. Anal., 25 (2022), 687–719.

[5] M. Y. Bhat and A. H. Dar, Convolution and Correlation Theorems for Wigner-Ville Distribution Associated with the Quaternion Offset Linear Canonical Transform, Sig. Image Video Proces., 16 (2022), 1235–1242.

[6] S. A. L. Collins, Lens-system diffraction integral written in terms of matrix optics, J. Opt. Soc. Am., 60 (1970), 1168–1177.

[7] M. Y. Bhat, Introductory Chapter: The Generalizations of the Fourier Transform, *Time Frequency Analysis of Some Generalized Fourier Transforms*, 1-13, IntechOpen, 2023.

[8] D. T. Haimo, Integral equations associated with Hankel convolution, Trans. Am. Math. Soc., 116 (1965), 330–375.

[9] I. I. Hirschman, Variation diminishing Hankel transform, J. Analyse Math., 8 (1960/61/1961), 307–336.

[10] M. Moshinsky and C. Quesne, Oscillator systems, In: Proceedings of the 15th Solvay Conference on Physics, 28 Sep.-3 Oct. 1970, New York: Gordon and Breach, pp. 233–300.

[11] R. S. Pathak and M. M. Dixit, Continuous and discrete Bessel wavelet transforms, J. Comput. Appl. Math. 160(2003), 241–250

[12] A. Prasad, A. Mahato and M. M. Dixit, Continuity of the Bessel wavelet transform on certain Beurling-type function spaces, J. Inequal. Appl., (2013), 2013:29, 9 pp.

[13] A. Prasad, A. Mahato, V. K. Singh and M. M. Dixit, The continuous fractional Bessel wavelet transformation, Bound. Value Prob., 40 (2013), 1–16.

[14] S. C. Pei, J. J. Ding, Eigenfunctions of linear canonical transform, IEEE Trans Signal Process., 50 (2002), no. 1, 11–26.

[15] S. K. Upadhyay, R. N. Yadav and L. Debnath, On continuous Bessel wavelet transformation associated with the Hankel-Hausdorff operator, Integral Transforms Spec. Funct., 23 (2012), no. 5, 315–323.

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