Research Article

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Wigner-Ville distribution and ambiguity function associated with the quaternion offset linear canonical transform

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Abstract: Wigner-Ville transform or Wigner-Ville distribution (WVD) associated with quaternion offset linear canonical transform (QOLCT) was proposed by Bhat and Dar. This work is devoted to the development of the theory proposed by them, which is an emerging tool in the scenario of signal processing. The main contribution of this work is to introduce WVD and ambiguity function (AF) associated with the QOLCT (WVD-QOLCT/AF-QOLCT). First, the definition of the WVD-QOLCT is proposed, and then several important properties such as dilation, nonlinearity, and boundedness are derived. Second, we derived the AF for the proposed transform. A bunch of important properties, including the reconstruction formula associated with the AF, are studied.

Keywords: quaternion algebra, quaternion offset linear canonical transform, Wigner-Ville distribution, ambiguity function, dilation

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1 Introduction

In the modern era of information theory, Wigner-Ville distribution (WVD), or more commonly Wigner-Ville transform (WVT), is an important and effective tool. It was the great Eugene Wigner who introduced the concept of WVD during the calculation of quantum corrections. Moving on the same road in 1948, Ville derived it independently by using the quadratic representation of a signal in the local time-frequency domain. The survey on this novel transform continued. This transform can be generalized to a linear canonical transform (LCT) by changing the kernel of the Fourier transform (FT) to that of an LCT in the WVT domain. For more about this transform, we refer to [1–11].

On the one hand, the quaternion Fourier transform (QFT) is worthwhile to explore in the era of communications. A bunch of needed properties like energy conservation, uncertainty principle, shift, modulation, differentiation, convolution, and correlation of QFT have been explored. Various number of transforms generalized from QFTs are closely related, for instance, the fractional QFT, quaternion wavelet transform, quaternion linear canonical transform (QLCT), and many more. Like the case of QFTs, one can also extend

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the WVD to the quaternion algebra while having similar properties to that in the classical sense. Many researchers generalized the WVD to the novel quaternion algebra, termed the quaternion WVD. One can see [12–19].

On the other hand, the ambiguity function (AF) that was introduced by Woodward in 1953 plays an important role in the Information theory. It has been applied in various fields, such as radar signal processing, sonar technology, and optical information theory (see [20–26]). In recent times, a lot of research work has been carried out by coupling the classical AF with other transforms. Other notable works are found in [27–30]. The WVD and AF associated with the LCT have great advantages and flexibility over the classical WVD and AF, respectively. The generalized WVD and AF have achieved better detection performance. Therefore, it is worthwhile to study a new generalized WVD and AF.

Recently, Bhat and Dar [31] have studied convolution and correlation theorems for WVD for the quaternion offset linear canonical transform (QOLCT). However, due to the noncommutativity of the multiplications in quaternion signals, only a special condition for the convolution and correlation theorems can be established. To be precise, we are bound to take real-valued signals in the function space over the Hamiltonian group. We continued their studies and established some more properties like dilation, nonlinearity, and boundedness. Nevertheless, to extend the scope of study, we introduced an ambiguity function for the QOLCT and were at ease to obtain the classical properties associated with it.

The article is organized as follows. In Section 2, we look at some basics about the quaternion algebra and the QOLCT. Section 3 is devoted to the definition of WVD associated with the QOLCT (WVD-QOLCT). Various properties of the WVD-QOLCT are investigated. In Section 4, we first introduce the AF for the QOLCT. We continued the study and obtained various properties like nonlinearity, boundedness, and a reconstruction formula for the proposed transform.

2 Preliminary

Here, we look at some basics needed for the rest of the article.

2.1 Quaternion algebra

The quaternion algebra is an extended version of complex numbers. Hamilton invented it in 1843, and in his honour, it is denoted by $\mathbb{H}$. The elements of $\mathbb{H}$ have the Cartesian form as

$$\mathbb{H} = \{q|q := [q]_0 + \mathbf{i}[q]_1 + \mathbf{j}[q]_2 + \mathbf{k}[q]_3,\ [q]_i \in \mathbb{R},\ i = 0, 1, 2, 3\},$$

where i, j, and k are imaginary units obeying the Hamilton's multiplication rules (see [18]),

$$ij = -ji = k, \quad jk = -kj = i,\ \ ki = -ik = j, \tag{1}$$

$$i^2 = j^2 = k^2 = ijk = -1. \tag{2}$$

Looking at (1), $\mathbb{H}$ is obviously noncommutative. So, we cannot directly extend a number of results on complex numbers to the quaternion. For the sake of convenience, we represent a quaternion q to be the sum of scalar q_1 and purely $3D$ quaternion q. Thus, we can write explicitly every quaternion as follows:

$$q = q_1 + iq_2 + jq_3 + kq_4 \in \mathbb{H}, \quad q_1, q_2, q_3, q_4 \in \mathbb{R}.$$

The conjugation is obtained by changing the sign of the pure part, i.e.,

$$\overline{q} = q_1 - iq_2 - jq_3 - kq_4.$$

It is evident that quaternion conjugation is linear but anti-involution,

$$\overline{\overline{p}} = p, \quad \overline{p+q} = \overline{p} + \overline{q}, \quad \overline{pq} = \overline{q}\,\overline{p},\ \forall p, q \in \mathbb{H}.$$

Modulus of a quaternion q can be defined as follows:

$$|q| = \sqrt{q\overline{q}} = \sqrt{q_1^2 + q_2^2 + q_3^2 + q_4^2}, \; |pq| = |p||q|.$$

It is evident that

$$|pq| = |p||q|, \; |q| = |\overline{q}|, \; p, q \in \mathbb{H}.$$

Any function $f : \mathbb{R}^2 \to \mathbb{H}$ can be viewed as

$$f(x, y) := f_1(x, y) + if_2(x, y) + jf_3(x, y) + kf_4(x, y),$$

where $(x, y) \in \mathbb{R} \times \mathbb{R}$.

The inner product of the functions f, g in $\mathbb{H}$ can be defined as follows:

$$\langle f, g\rangle_{L^2(\mathbb{R}^2,\mathbb{H})} = \int_{\mathbb{R}^2} f(\mathbf{x})\overline{g(\mathbf{x})}d^2\mathbf{x}. \tag{3}$$

If we let $f = g$, then

$$\|f\|_2^2 = \langle f, f\rangle_{L^2(\mathbb{R}^2,\mathbb{H})} = \int_{\mathbb{R}^2} |f(\mathbf{x})|^2 d^2\mathbf{x}. \tag{4}$$

The space $L^2(\mathbb{R}^2, \mathbb{H})$ can be defined like

$$L^2(\mathbb{R}^2, \mathbb{H}) = \{f | f : \mathbb{R}^2 \to \mathbb{H}, \|f\|_2 < \infty\}. \tag{5}$$

Lemma 1. *If $f, g \in L^2(\mathbb{R}^2, H)$, then the Cauchy-Schwarz inequality* [15] *holds*

$$|\langle f, g\rangle_{L^2(\mathbb{R}^2,\mathbb{H})}|^2 \le \|f\|^2_{L^2(\mathbb{R}^2,\mathbb{H})}\|g\|^2_{L^2(\mathbb{R}^2,\mathbb{H})} \tag{6}$$

if and only if $f = \lambda g$ for some quaternionic parameter $\lambda \in \mathbb{H}$, the equality holds.

2.2 QLCT

QLCT was first of all defined by Morais et al. [19]. They have considered two matrixes with real values as

$$A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in \mathbb{R}^{2\times 2},$$

with $a_1d_1 - b_1c_1 = 1$ and $a_2d_2 - b_2c_2 = 1$. Hitzer in [16] generalized the definitions of [19] into the case of two-sided QLCT of signals $f \in L^1(\mathbb{R}^2, \mathbb{H})$ like

$$\mathcal{L}^{i,j}_{A_1,A_2}[f](\mathbf{u}) = \begin{cases} \int_{\mathbb{R}^2} K^i_{A_1}(t_1, u_1)f(\mathbf{t})K^j_{A_2}(t_2, u_2)\mathrm{d}^2\mathbf{t}, & b_1, b_2 \neq 0; \\ \sqrt{d_1}\, e^{i\frac{c_1d_1}{2}{u_1}^2} f(d_1u_1, t_2)K^j_{A_2}(t_2, u_2), & b_1 = 0, b_2 \neq 0; \\ \sqrt{d_2}\, K^i_{A_1}(t_1, u_1)f(t_1, d_2u_2)e^{j\frac{c_2d_2}{2}u_2^2}, & b_1 \neq 0, b_2 = 0; \\ \sqrt{d_1d_2}\, e^{i\frac{c_1d_1}{2}{u_1}^2} f(d_1u_1, d_2u_2)e^{j\frac{c_2d_2}{2}u_2^2}, & b_1 = b_2 = 0, \end{cases} \tag{7}$$

where $\mathbf{t} = (t_1, t_2)$ and $\mathbf{u} = (u_1, u_2)$.

Theorem 1. (Inversion formula for QLCT) *Suppose $f \in L(\mathbb{R}^2, \mathbb{H})$, then the inversion of the two-sided QLCT of f is given as follows:*

$$f(\mathbf{x}) = \left\{\mathcal{L}^{-1}_{A_1,A_2}\right\}\left\{\mathcal{L}^{\mathbb{H}}_{A_1,A_2}[f]\right\}(\mathbf{x}) = \int_{\mathbb{R}^2} K^{-i}_{A_1}(x_1, u_1)\mathcal{L}^{\mathbb{H}}_{A_1,A_2}[f](\mathbf{u})K^{-j}_{A_2}(x_2, u_2)\mathrm{d}^2\mathbf{u}. \tag{8}$$

Theorem 2. (Plancherel for QLCT) *Every 2D quaternion-valued signal $f \in L^2(\mathbb{R}^2, \mathbb{H})$, and its QLCT are related to the Placherel identity in the following way*:

$$\|f\|_{L^2(\mathbb{R}^2,\mathbb{H})} = \|\mathcal{L}^{\mathbb{H}}_{A_1,A_2}[f]\|_{Q,2}. \tag{9}$$

2.3 QOLCT

Amid the noncommutative nature in the multiplication of quaternions, we have three types of the QOLCT. The first one is left-sided QOLCT, the second one is right-sided QOLCT, and the third one is two-sided QOLCT.

Definition 1. Let $A_i = \begin{bmatrix} a_i & b_i & | & r_i \\ c_i & d_i & | & s_i \end{bmatrix}$ be a matrix parameter such that $a_i, b_i, c_i, d_i, r_i, s_i \in \mathbb{R}$, and $a_id_i - b_ic_i = 1$ for $i = 1, 2$. The two-sided QOLCT of any quaternion-valued function $f \in L^2(\mathbb{R}^2, \mathbb{H})$ is given by

$$O^{i,j}_{A_1,A_2}[f(\mathbf{t})](\mathbf{u}) = \begin{cases} \int_{\mathbb{R}^2} K^i_{A_1}(t_1, u_1)f(\mathbf{t})K^j_{A_2}(t_2, u_2)\mathrm{d}^2\mathbf{t}, & b_1, b_2 \neq 0, \\ \sqrt{d_1}\, e^{i\left[\frac{c_1d_1}{2}(u_1-r_1)^2+u_1r_1\right]} f(d_1u_1 - d_1r_1, t_2)K^j_{A_2}(t_2, u_2), & b_1 = 0, b_2 \neq 0, \\ \sqrt{d_2}K^i_{A_1}(t_1, u_1)f(t_1, d_2u_2 - d_2r_2)e^{j\left[\frac{c_2d_2}{2}(u_2-r_2)^2+u_2r_2\right]}, & b_1 \neq 0, b_2 = 0, \\ \sqrt{d_1d_2}\, f(d_1u_1 - d_1r_1, d_2u_2 - d_2r_2)e^{i\left[\frac{c_1d_1}{2}(u_1-r_1)^2+u_1r_1\right]}e^{j\left[\frac{c_2d_2}{2}(u_2-r_2)^2+u_2r_2\right]}, & b_1 = 0, b_2 = 0, \end{cases} \tag{10}$$

where $\mathbf{t} = (t_1, t_2)$ and $\mathbf{u} = (u_1, u_2)$, and the quaternion kernels $K^i_{A_1}(t_1, u_1)$ and $K^j_{A_2}(t_2, u_2)$, are respectively, given as follows:

$$K^i_{A_1}(t_1, u_1) = \frac{1}{\sqrt{2\pi b_1 i}} e^{\frac{i}{2b_1}[a_1t_1^2 - 2t_1(u_1-r_1) - 2u_1(d_1r_1 - b_1s_1) + d_1(u_1^2+r_1^2)]}, \quad b_1 \neq 0 \tag{11}$$

$$K^j_{A_2}(t_2, u_2) = \frac{1}{\sqrt{2\pi b_2 j}} e^{\frac{j}{2b_2}[a_2t_2^2 - 2t_2(u_2-r_2) - 2u_2(d_2r_2 - b_2s_2) + d_2(u_2^2+r_2^2)]}, \quad b_2 \neq 0. \tag{12}$$

Remark 1. One can define the left- and right-sided QOLCT by simply placing the two above kernels both on the left or on the right.

Note that, when $r_1 = r_2 = s_1 = s_2 = 0$, the two-sided QOLCT reduces to the QLCT.

Also, when $A_1 = A_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$, the conventional two-sided QFT is recovered.

Theorem 3. *Suppose $f \in L^2(\mathbb{R}^2, \mathbb{H})$, then the inversion of two-sided QOLCT is given by*

$$f(\mathbf{t}) = \int_{\mathbb{R}^2} K^{-i}_{A_1}(t_1, u_1)O^{i,j}_{A_1,A_2}[f](\mathbf{u})K^{-j}_{A_2}(t_2, u_2)d^2\mathbf{u}. \tag{13}$$

Theorem 4. (Plancherel for QOLCT) *Every two-dimensional quaternion-valued function $f \in L^2(\mathbb{R}^2, \mathbb{H})$ and its two-sided QOLCT are related to the Plancherel identity in the following way*:

$$\|O^{i,j}_{A_1,A_2}[f]\|_{L^2(\mathbb{R}^2,\mathbb{H})} = \|f\|_{L^2(\mathbb{R}^2,\mathbb{H})}. \tag{14}$$

3 WVD-QOLCT

Recently, authors in [31] defined WVD-QOLCT and studied some properties associated with it and also established a special condition for the convolution and correlation theorems for the WVD-QQPFT, which are important for signal processing. In this section, we continue to study the properties of WVD-QOLCT, and prior to that, we first define the WVD-QOLCT.

Definition 2. Let $A_i = \begin{bmatrix} a_i & b_i & | & r_i \\ c_s & d_i & | & s_i \end{bmatrix}$ be a matrix parameter such that $a_s, b_i, c_i, d_i, r_i, s_i \in \mathbb{R}$, and $a_i d_i - b_i c_i = 1$, for $i = 1, 2$. The WVD associated with the two-sided QOLCT (WVD-QOLCT) of signals $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$ is given as follows:

$$\mathcal{W}^{A_1,A_2}_{f,g}(\mathbf{t}, \mathbf{w}) = \int_{\mathbb{R}^2} K^i_{A_1}(z_1, w_1) f\left(\mathbf{t} + \frac{\mathbf{z}}{2}\right)\overline{g\left(\mathbf{t} - \frac{\mathbf{z}}{2}\right)} K^j_{A_2}(z_2, w_2)\mathrm{d}^2\mathbf{z}, \quad b_i \neq 0, \tag{15}$$

where $\mathbf{t} = (t_1, t_2)$, $\mathbf{w} = (w_1, w_2)$, $\mathbf{z} = (z_1, z_2)$, and the quaternion kernels $K^i_{A_1}(z_1, w_1)$ and $K^j_{A_2}(z_2, w_2)$ are given by (11) and (12), respectively.

Note: If $f = g$, then $\mathcal{W}^{A_1,A_2}_{f,f}(\mathbf{t}, \mathbf{w})$ we call it the Auto WVD-QOLCT. Otherwise, it is labelled as the cross WVD-QOLCT.

In this article, we always assume that $b_i \neq 0$, $i = 1, 2$, otherwise the proposed transform reduces to a chirp multiplication. Hence, for any $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$, we have,

$$\begin{aligned} \mathcal{W}^{A_1,A_2}_{f,g}(\mathbf{t}, \mathbf{w}) &= \int_{\mathbb{R}^2} K^i_{A_1}(z_1, w_1) f\left(\mathbf{t} + \frac{\mathbf{z}}{2}\right)\overline{g\left(\mathbf{t} - \frac{\mathbf{z}}{2}\right)} K^j_{A_2}(z_2, w_2)\mathrm{d}^2\mathbf{w} \\ &= O^{i,j}_{A_1,A_2}\left\{ f\left(\mathbf{t} + \frac{\mathbf{z}}{2}\right)\overline{g\left(\mathbf{t} - \frac{\mathbf{z}}{2}\right)}\right\} \\ &= O^{i,j}_{A_1,A_2}\{h_{f,g}(\mathbf{t}, \mathbf{w})\}, \end{aligned} \tag{16}$$

where $h_{f,g}(\mathbf{t}, \mathbf{w}) = f\left(\mathbf{t} + \frac{\mathbf{z}}{2}\right)\overline{g\left(\mathbf{t} - \frac{\mathbf{z}}{2}\right)}$ is known as quaternion correlation product.

Applying the inverse QOLCT to (16), we obtain

$$h_{f,g}(\mathbf{t}, \mathbf{z}) = \left\{O^{i,j}_{A_1,A_2}\right\}^{-1}\left\{\mathcal{W}^{A_1,A_2}_{f,g}(\mathbf{t}, \mathbf{w})\right\}, \tag{17}$$

which implies

$$\begin{aligned} f\left(\mathbf{t} + \frac{\mathbf{z}}{2}\right)\overline{g\left(\mathbf{t} - \frac{\mathbf{z}}{2}\right)} &= \left\{O^{i,j}_{A_1,A_2}\right\}^{-1}\left\{\mathcal{W}^{A_1,A_2}_{f,g}(\mathbf{t}, \mathbf{w})\right\} \\ &= \int_{\mathbb{R}^2} K^{-i}_{A_1}(t_1, w_1)\mathcal{W}^{A_1,A_2}_{f,g}(\mathbf{t}, \mathbf{w}) K^{-j}_{A_2}(t_2, w_2)\mathrm{d}^2\mathbf{w}. \end{aligned} \tag{18}$$

We now study several important properties of the WVD-QOLCT defined by (2).

Theorem 5. (Dilation) *Let $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$, then*

$$\mathcal{W}^{A_1,A_2}_{\mathcal{D}_m f, \mathcal{D}_m g}(\mathbf{t}, \mathbf{w}) = \mathcal{W}^{B_1,B_2}_{f,g}\left(\frac{\mathbf{t}}{m}, \frac{\mathbf{w}}{m}\right), \tag{19}$$

where $A_i = \begin{bmatrix} a_i & b_i & | & r_i \\ c_s & d_i & | & s_i \end{bmatrix}$, $B_i = \begin{bmatrix} a_i & \frac{b_i}{m^2} & | & \frac{r_i}{m} \\ m^2 c_s & d_i & | & m s_i \end{bmatrix}$, $i = 1, 2$, and $\mathcal{D}_m f(\mathbf{t}) = \frac{1}{m} f\left(\frac{\mathbf{t}}{m}\right)$, $m \neq 0$.

Proof. We have from Definition 2

$$
\begin{aligned}
\mathcal{W}^{A_1,A_2}_{\mathcal{D}_m f,\mathcal{D}_m g}(\mathbf{t},\mathbf{w}) &= \frac{1}{m^2}\int_{\mathbb{R}^2} K^i_{A_1}(z_1,w_1) f\left(\frac{\mathbf{t}}{m}+\frac{\mathbf{z}}{2m}\right)\overline{g\left(\frac{\mathbf{t}}{m}-\frac{\mathbf{z}}{2m}\right)} K^j_{A_2}(z_2,w_2)\,\mathrm{d}^2\mathbf{z} \\
&= \int_{\mathbb{R}^2} K^i_{A_1}(mx_1,w_1) f\left(\frac{\mathbf{t}}{m}+\frac{\mathbf{x}}{2}\right)\overline{g\left(\frac{\mathbf{t}}{m}-\frac{\mathbf{x}}{2}\right)} K^j_{A_2}(mx_2,w_2)\,\mathrm{d}^2\mathbf{x} \\
&= \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1 i}} e^{\frac{i}{2b_1}[a_1 m^2 x_1^2 - 2mx_1(w_1-r_1) - 2w_1(d_1 r_1 - b_1 s_1) + d_1(w_1^2+r_1^2)]} f\left(\frac{\mathbf{t}}{m}+\frac{\mathbf{z}}{2m}\right)\overline{g\left(\frac{\mathbf{t}}{m}-\frac{\mathbf{z}}{2m}\right)} \\
&\quad \times \frac{1}{\sqrt{2\pi b_2 j}} e^{\frac{j}{2b_2}[a_2 m^2 x_2^2 - 2mx_2(w_2-r_2) - 2w_2(d_2 r_2 - b_2 s_2) + d_2(w_2^2+r_2^2)]} d^2\mathbf{x} \\
&= \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1 i}} e^{\frac{i}{2b_1}\left[\frac{a_1}{\frac{1}{m^2}} x_1^2 - 2\frac{1}{\frac{1}{m^2}} x_1\left(\frac{w_1}{m}-\frac{r_1}{m}\right) - \frac{2}{\frac{1}{m^2}}\frac{w_1}{m}\left(d_1\frac{r_1}{m} - \frac{b_1}{m^2} m s_1\right) + \frac{d_1}{\frac{1}{m^2}}\left(\left(\frac{w_1}{m}\right)^2 + \left(\frac{r_1}{m}\right)^2\right)\right]} f\left(\frac{\mathbf{t}}{m}+\frac{\mathbf{z}}{2m}\right) \\
&\quad \times \overline{g\left(\frac{\mathbf{t}}{m}-\frac{\mathbf{z}}{2m}\right)} \frac{1}{\sqrt{2\pi b_2 j}} e^{\frac{j}{2b_2}\left[\frac{a_2}{\frac{1}{m^2}} x_2^2 - 2\frac{1}{\frac{1}{m^2}} x_2\left(\frac{w_2}{m}-\frac{r_2}{m}\right) - \frac{2}{\frac{1}{m^2}}\frac{w_2}{m}\left(d_2\frac{r_2}{m} - \frac{b_2}{m^2} m s_2\right) + \frac{d_2}{\frac{1}{m^2}}\left(\left(\frac{w_2}{m}\right)^2 + \left(\frac{r_2}{m}\right)^2\right)\right]} d^2\mathbf{x} \\
&= \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1 i}} e^{\frac{i}{2\frac{b_1}{m^2}}\left[a_1 x_1^2 - 2x_1\left(\frac{w_1}{m}-\frac{r_1}{m}\right) - 2\frac{w_1}{m}\left(d_1\frac{r_1}{m} - \frac{b_1}{m^2} m s_1\right) + d_1\left(\left(\frac{w_1}{m}\right)^2 + \left(\frac{r_1}{m}\right)^2\right)\right]} f\left(\frac{\mathbf{t}}{m}+\frac{\mathbf{z}}{2m}\right) \\
&\quad \times \overline{g\left(\frac{\mathbf{t}}{m}-\frac{\mathbf{z}}{2m}\right)} \frac{1}{\sqrt{2\pi b_2 j}} e^{\frac{j}{2\frac{b_2}{m^2}}\left[a_2 x_2^2 - 2x_2\left(\frac{w_2}{m}-\frac{r_2}{m}\right) - 2\frac{w_2}{m}\left(d_2\frac{r_2}{m} - \frac{b_2}{m^2} m s_2\right) + d_2\left(\left(\frac{w_2}{m}\right)^2 + \left(\frac{r_2}{m}\right)^2\right)\right]} d^2\mathbf{x} \\
&= \int_{\mathbb{R}^2} K^i_{B_1}\left(x_1,\frac{w_1}{m}\right) f\left(\frac{\mathbf{t}}{m}+\frac{\mathbf{z}}{2m}\right)\overline{g\left(\frac{\mathbf{t}}{m}-\frac{\mathbf{z}}{2m}\right)} K^j_{B_2}\left(x_2,\frac{w_2}{m}\right)\mathrm{d}^2\mathbf{x} \\
&= \mathcal{W}^{B_1,B_2}_{f,g}\left(\frac{\mathbf{t}}{m},\frac{\mathbf{w}}{m}\right), \quad \text{where } B_i = \begin{bmatrix} a_i & \frac{b_i}{m^2} & | & \frac{r_i}{m} \\ m^2 c_s & d_i & | & m s_i \end{bmatrix}, \quad i = 1, 2,
\end{aligned}
$$

which completes the proof. □

Theorem 6. (Shift) *Let $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$, then*

$$
\mathcal{W}^{A_1,A_2}_{T_m f, T_m g}(\mathbf{t},\mathbf{w}) = \mathcal{W}^{A_1,A_2}_{f,g}(\mathbf{t}-\mathbf{m},\mathbf{w}), \tag{20}
$$

where $T_{\mathbf{m}} f(\mathbf{t}) = f(\mathbf{t}-\mathbf{m})$, $\quad \mathbf{t} = (t_1, t_2)$.

Proof. From Definition 2, we have

$$
\begin{aligned}
\mathcal{W}^{A_1,A_2}_{T_{\mathbf{m}} f, T_{\mathbf{m}} g}(\mathbf{t},\mathbf{w}) &= \int_{\mathbb{R}^2} K^i_{A_1}(z_1,w_1) f\left(\mathbf{t}+\frac{\mathbf{z}}{2}-\mathbf{m}\right)\overline{g\left(\mathbf{t}-\frac{\mathbf{z}}{2}-\mathbf{m}\right)} K^j_{A_2}(z_2,w_2)\,\mathrm{d}^2\mathbf{z} \\
&= \int_{\mathbb{R}^2} K^i_{A_1}(z_1,w_1) f\left((\mathbf{t}-\mathbf{m})+\frac{\mathbf{z}}{2}\right)\overline{g\left((\mathbf{t}-\mathbf{m})-\frac{\mathbf{z}}{2}\right)} K^j_{A_2}(z_2,w_2)\,\mathrm{d}^2\mathbf{z} \\
&= \mathcal{W}^{A_1,A_2}_{f,g}(\mathbf{t}-\mathbf{m},\mathbf{w}),
\end{aligned}
$$

which completes the theorem. □

Theorem 7. (Boundedness) *Let $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$. Then,*

$$
|\mathcal{W}^{A_1,A_2}_{f,g}(\mathbf{t},\mathbf{w})| \le \frac{2}{\pi\sqrt{b_1 b_2}}\|f\|_{L^2(\mathbb{R}^2,H)}\|g\|_{L^2(\mathbb{R}^2,H)}. \tag{21}
$$

Proof. With the help of Cauchy-Schwarz inequality for quaternion domain (6), we obtain

$$
\begin{aligned}
|\mathcal{W}_{f,g}^{A_1,A_2}(\mathbf{t},\mathbf{w})|^2 &= \left| \int_{\mathbb{R}^2} K_{A_1}^{i}(z_1,w_1)\, f\left(t+\frac{z}{2}\right)\overline{g\left(\mathbf{t}-\frac{\mathbf{z}}{2}\right)}K_{A_2}^{j}(z_2,w_2)\mathrm{d}n \right|^2 \\
&\leq \left(\int_{\mathbb{R}^2} \left| K_{A_1}^{\mathbf{i}}(z_1,w_1) f\left(\mathbf{t}+\frac{\mathbf{z}}{2}\right)\overline{g\left(\mathbf{t}-\frac{\mathbf{z}}{2}\right)}K_{A_2}^{\mathbf{j}}(z_2,w_2)\right| d^2\mathbf{z} \right)^2 \\
&= \left(\frac{1}{\sqrt{4\pi^2|b_1b_2|}} \int_{\mathbb{R}^2} \left| f\left(\mathbf{t}+\frac{\mathbf{z}}{2}\right)\overline{g\left(\mathbf{t}-\frac{\mathbf{z}}{2}\right)} \right| d^2\mathbf{z} \right)^2 \\
&\leq \frac{1}{4\pi^2|b_1b_2|}\left(\int_{\mathbb{R}^2} \left| f\left(\mathbf{t}+\frac{\mathbf{z}}{2}\right)\right|^2 d^2\mathbf{z}\right)\left(\int_{\mathbb{R}^2} \left| \overline{g\left(\mathbf{t}-\frac{\mathbf{z}}{2}\right)}\right|^2 d^2\mathbf{z}\right) \\
&= \frac{1}{4\pi^2|b_1b_2|}\left(4\int_{\mathbb{R}^2} |f(\mathbf{w})|^2 d^2\mathbf{w}\right)\left(4\int_{\mathbb{R}^2} |\overline{g(\mathbf{y})}|^2 d^2\mathbf{y}\right) \\
&= \frac{4}{\pi^2|b_1b_2|}\|f\|^2_{L^2(\mathbb{R}^2,H)}\|g\|^2_{L^2(\mathbb{R}^2,H)},
\end{aligned}
$$

with the changing of variables $\mathbf{w} = \mathbf{t} + \frac{\mathbf{z}}{2}$ and $\mathbf{y} = \mathbf{t} - \frac{\mathbf{z}}{2}$ in last second step, we obtain

$$
|\mathcal{W}_{f,g}^{A_1,A_2}(\mathbf{t},\mathbf{w})| \leq \frac{2}{\pi\sqrt{|b_1b_2|}}\|f\|_{L^2(\mathbb{R}^2,H)}\|g\|_{L^2(\mathbb{R}^2,H)},
$$

which completes the proof. □

Theorem 8. (Nonlinearity) *Let f and g be two quaternion functions in $L^2(\mathbb{R}^2,\mathbb{H})$. Then,*

$$
\mathcal{W}_{f+g}^{A_1,A_2} = \mathcal{W}_{f,f}^{A_1,A_2} + \mathcal{W}_{f,g}^{A_1,A_2} + \mathcal{W}_{g,f}^{A_1,A_2} + \mathcal{W}_{g,g}^{A_1,A_2}. \tag{22}
$$

Proof. By Definition 2, we have

$$
\begin{aligned}
\mathcal{W}_{f+g}^{A_1,A_2}(\mathbf{t},\mathbf{w}) &= \int_{\mathbb{R}^2} K_{A_1}^{i}(z_1,w_1)\left[f\left(\mathbf{t}+\frac{\mathbf{z}}{2}\right) + g\left(\mathbf{t}+\frac{\mathbf{z}}{2}\right)\right]\overline{\left[f\left(\mathbf{t}-\frac{\mathbf{z}}{2}\right) + g\left(\mathbf{t}-\frac{\mathbf{z}}{2}\right)\right]}K_{A_2}^{j}(z_2,w_2)\mathrm{d}^2\mathbf{z} \\
&= \int_{\mathbb{R}^2} K_{A_1}^{i}(z_1,w_1)\left[f\left(\mathbf{t}+\frac{\mathbf{z}}{2}\right)\overline{f\left(\mathbf{t}-\frac{\mathbf{z}}{2}\right)} + f\left(\mathbf{t}+\frac{\mathbf{z}}{2}\right)\overline{g\left(\mathbf{t}-\frac{\mathbf{z}}{2}\right)} + g\left(\mathbf{t}+\frac{\mathbf{z}}{2}\right)\overline{f\left(\mathbf{t}-\frac{\mathbf{z}}{2}\right)}\right. \\
&\qquad \left. + g\left(\mathbf{t}+\frac{\mathbf{z}}{2}\right)\overline{g\left(\mathbf{t}-\frac{\mathbf{z}}{2}\right)}\right]K_{A_2}^{j}(z_2,w_2)\mathrm{d}^2\mathbf{z} \\
&= \int_{\mathbb{R}^2} K_{A_1}^{i}(z_1,w_1) f\left(\mathbf{t}+\frac{\mathbf{z}}{2}\right)\overline{f\left(\mathbf{t}-\frac{\mathbf{z}}{2}\right)}K_{A_2}^{j}(z_2,w_2)\mathrm{d}^2\mathbf{z} + \int_{\mathbb{R}^2} K_{A_1}^{i}(z_1,w_1) f\left(\mathbf{t}+\frac{\mathbf{z}}{2}\right)\overline{g\left(\mathbf{t}-\frac{\mathbf{z}}{2}\right)}K_{A_2}^{j}(z_2,w_2)\mathrm{d}^2\mathbf{z} \\
&\quad + \int_{\mathbb{R}^2} K_{A_1}^{i}(z_1,w_1)\, g\left(\mathbf{t}+\frac{\mathbf{z}}{2}\right)\overline{\mathbf{f}\left(\mathbf{t}-\frac{\mathbf{z}}{2}\right)}K_{A_2}^{j}(z_2,w_2)\mathrm{d}^2\mathbf{z} \\
&\quad + \int_{\mathbb{R}^2} K_{A_1}^{i}(z_1,w_1)\, g\left(\mathbf{t}+\frac{\mathbf{z}}{2}\right)\overline{g\left(\mathbf{t}-\frac{\mathbf{z}}{2}\right)}K_{A_2}^{j}(z_2,w_2)\mathrm{d}^2\mathbf{z} \\
&= \mathcal{W}_{f,f}^{A_1,A_2} + \mathcal{W}_{f,g}^{A_1,A_2} + \mathcal{W}_{g,f}^{A_1,A_2} + \mathcal{W}_{g,g}^{A_1,A_2},
\end{aligned}
$$

which completes the proof. □

Theorem 9. *If* $\|f\|_{(\mathbb{R}^2,\mathbb{H})} = \|g\|_{(\mathbb{R}^2,\mathbb{H})} = 1$, *then we have,*

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \|\mathcal{W}_{f,g}^{A_1,A_2}(\mathbf{t},\mathbf{w})\|^2 d^2\mathbf{t}d^2\mathbf{w} = 4. \tag{23}$$

Proof. As $\mathcal{W}_{f,g}^{A_1,A_2}(\mathbf{t},\mathbf{w})$ is the QOLCT of $h_{f,g}(\mathbf{t},\mathbf{z})$, then by Plancherel for QOLCT (14), we obtain

$$\int_{\mathbb{R}^2} |\mathcal{W}_{f,g}^{A_1,A_2}(\mathbf{t},\mathbf{w})|^2 d^2\mathbf{w} = \int_{\mathbb{R}^2} |h_{f,g}^{A_1,A_2}(\mathbf{t},\mathbf{z})|^2 d^2\mathbf{z} = \int_{\mathbb{R}^2} \left| f\left(\mathbf{t}+\frac{\mathbf{z}}{2}\right)\overline{g\left(\mathbf{t}-\frac{\mathbf{z}}{2}\right)}\right|^2 d^2\mathbf{z}.$$

Setting $\mathbf{x} = \mathbf{t} + \frac{\mathbf{z}}{2}$ and $y = \mathbf{t} - \frac{\mathbf{z}}{2}$ and integrating the above equation with respect to $d\mathbf{t}$, we have

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\mathcal{W}_{f,g}^{A_1,A_2}(\mathbf{t},\mathbf{w})|^2 d^2\mathbf{w} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left| f\left(\mathbf{t}+\frac{\mathbf{z}}{2}\right)\overline{g\left(\mathbf{t}-\frac{\mathbf{z}}{2}\right)}\right|^2 d^2\mathbf{t}d\mathbf{z} = 4\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |f(\mathbf{x})|^2 \overline{|g(\mathbf{y})|^2}\, d^2\mathbf{y}d\mathbf{t},$$

which completes the proof. □

Theorem 10. (Reconstruction formula) *For* $f, g \in L^2(\mathbb{R}^2,\mathbb{H})$, *where* g *does not vanish at* 0, *we obtain the following inversion formula of the WVD-QOLCT*:

$$f(t) = \frac{1}{\overline{g(\mathbf{0})}} \int_{\mathbb{R}^2} K_{A_1}^{-i}(w_1,z_1)\mathcal{W}_{f,g}^{A_1,A_2}\left(\frac{\mathbf{t}}{2},\mathbf{u}\right)K_{A_2}^{-j}(w_2,z_2)\mathrm{d}^2\mathbf{w}. \tag{24}$$

Proof. See [31]. □

Theorem 11. (Orthogonality relation) *If* $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^2,\mathbb{H})$ *are quaternion-valued signals. Then,*

$$\langle \mathcal{W}_{f_1,g_1}^{A_1,A_2}(\mathbf{t},\mathbf{w}), \mathcal{W}_{f_2,g_2}^{A_1,A_2}(\mathbf{t},\mathbf{w})\rangle = [\langle f_1, f_2\rangle\langle g_2, g_1\rangle]_{\mathbb{H}}. \tag{25}$$

Proof. See [31]. □

Consequences of Theorem 11:

(i) If $g_1 = g_2 = g$, then

$$\langle \mathcal{W}_{f_1,g}^{A_1,A_2}(\mathbf{t},\mathbf{w}), \mathcal{W}_{f_2,g}^{A_1,A_2}(\mathbf{t},\mathbf{w})\rangle = \|g\|^2_{L^2(\mathbb{R}^2)}\langle f_1, f_2\rangle. \tag{26}$$

(ii) If $f_1 = f_2 = f$, then

$$\langle \mathcal{W}_{f,g_1}^{A_1,A_2}(\mathbf{t},\mathbf{w}), \mathcal{W}_{f,g_2}^{A_1,A_2}(\mathbf{t},\mathbf{w})\rangle = \|f\|^2_{L^2(\mathbb{R}^2)}\langle g_1, g_2\rangle. \tag{27}$$

(iii) If $f_1 = f_2 = f$ and $g_1 = g_2 = g$, then

$$\langle \mathcal{W}_{f,g}^{A_1,A_2}(\mathbf{t},\mathbf{w}), \mathcal{W}_{f,g}^{A_1,A_2}(\mathbf{t},\mathbf{w})\rangle = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\mathcal{W}_{f,g}^{A_1,A_2}(\mathbf{t},\mathbf{w})|^2 d^2\mathbf{w}d^2\mathbf{t} = \|f\|^2_{L^2(\mathbb{R}^2)}\|g\|^2_{L^2(\mathbb{R}^2)}. \tag{28}$$

Equation (28) represents the Plancherel theorem for WVD-QOLCT.

Now, we move forward towards next section that is AF associated with QOLCT and its properties.

4 AF associated with QOLCT

Wood ward in 1953 introduced the classical AF for mathematical analysis of sonar and radar signals. Recently, authors in [32] introduced the AF associated with QLCT (AF-QLCT). In this section, we generalize the AF-QLCT to the offset linear canonical domain.

Definition 3. Let $A_i = \begin{bmatrix} a_i & b_i & | & r_i \\ c_s & d_i & | & s_i \end{bmatrix}$ be a matrix parameter such that $a_s,\ b_i,\ c_i,\ d_i,\ r_i,\ s_i \in \mathbb{R}$, and $a_i d_i - b_i c_i = 1$, for $i = 1, 2$. The AF associated with the two-sided QOLCT (AF-QOLCT) of signals $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$, is given by

$$\mathcal{A}_{f,g}^{A_1,A_2}(\mathbf{t}, \mathbf{w}) = \int_{\mathbb{R}^2} K_{A_1}^i(z_1, w_1) f\left(\mathbf{z} + \frac{\mathbf{t}}{2}\right) \overline{g\left(\mathbf{z} - \frac{t}{2}\right)} K_{A_1}^j(z_2, w_2) \mathrm{d}^2\mathbf{z}. \tag{29}$$

Theorem 12. *Let $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$, then*

$$\mathcal{A}_{f(-\mathbf{t})}^{A_1,A_2}(\mathbf{t}, \mathbf{w}) = \mathcal{A}_{f(\mathbf{t})}^{B_1,B_2}(\mathbf{t}, -\mathbf{w}), \tag{30}$$

where $B_i = \begin{bmatrix} a_i & b_i & | & -r_i \\ c_s & d_i & | & -s_i \end{bmatrix}$ *for* $i = 1, 2$.

Proof. From (29), we have

$$\begin{aligned}
\mathcal{A}_{f(-\mathbf{t})}^{A_1,A_2}(\mathbf{t}, \mathbf{w}) &= \int_{\mathbb{R}^2} K_{A_1}^i(z_1, w_1) f\left(-\mathbf{z} + \frac{\mathbf{t}}{2}\right) \overline{g\left(-\mathbf{z} - \frac{\mathbf{t}}{2}\right)} K_{A_1}^j(z_2, w_2) \mathrm{d}^2\mathbf{z} \\
&= \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi i}} e^{\frac{i}{2b_1}[a_1 z_1^2 - 2z_1(w_1 - r_1) - 2w_1(d_1 r_1 - b_1 s_1) + d_1(w_1^2 + r_1^2)]} f \\
&\quad \times \left(-\mathbf{z} + \frac{\mathbf{t}}{2}\right) \overline{g\left(-\mathbf{z} - \frac{\mathbf{t}}{2}\right)} \frac{1}{\sqrt{2\pi j}} e^{\frac{j}{2b_2}[a_2 z_2^2 - 2z_2(w_2 - r_2) - 2w_2(d_2 r_2 - b_2 s_2) + d_2(w_2^2 + r_2^2)]} d^2\mathbf{z}.
\end{aligned}$$

On setting $-\mathbf{z} = \mathbf{z}'$, we have from the above equation

$$\begin{aligned}
\mathcal{A}_{f(-\mathbf{t})}^{A_1,A_2}(\mathbf{t}, \mathbf{w}) &= \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi i}} e^{\frac{i}{2b_1}[a_1(-z_1')^2 - 2(-z_1')(w_1 - r_1) - 2w_1(d_1 r_1 - b_1 s_1) + d_1(w_1^2 + r_1^2)]} f\left(\mathbf{z}' + \frac{\mathbf{t}}{2}\right) \overline{g\left(\mathbf{z}' - \frac{\mathbf{t}}{2}\right)} \\
&\quad \times \frac{1}{\sqrt{2\pi j}} e^{\frac{j}{2b_2}[a_2(-z_2')^2 - 2(-z_2')(w_2 - r_2) - 2w_2(d_2 r_2 - b_2 s_2) + d_2(w_2^2 + r_2^2)]} d^2\mathbf{z}' \\
&= \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi i}} e^{\frac{i}{2b_1}[a_1 z_1'^2 - 2z_1'(-w_1 - (-r_1)) - 2(-w_1)(d_1(-r_1) - b_1(-s_1)) + d_1((-w_1)^2 + (-r_1)^2)]} f\left(\mathbf{z}' + \frac{\mathbf{t}}{2}\right) \overline{g\left(\mathbf{z}' - \frac{\mathbf{t}}{2}\right)} \\
&\quad \times \frac{1}{\sqrt{2\pi j}} e^{\frac{j}{2b_2}[a_2 z_2'^2 - 2z_2'(-w_2 - (-r_2)) - 2(-w_2)(d_2(-r_2) - b_2(-s_2)) + d_2((-w_2)^2 + (-r_2)^2)]} d^2\mathbf{z}' \\
&= \int_{\mathbb{R}^2} K_{B_1}^i(z_1', -w_1) f\left(\mathbf{z}' + \frac{\mathbf{t}}{2}\right) \overline{g\left(\mathbf{z}' - \frac{\mathbf{t}}{2}\right)} K_{B_1}^j(z_2', -w_2) \mathrm{d}^2\mathbf{z}' \\
&= \mathcal{A}_{f(\mathbf{t})}^{B_1,B_2}(\mathbf{t}, -\mathbf{w}), \quad \text{where } B_i = \begin{bmatrix} a_i & b_i & | & -r_i \\ c_s & d_i & | & -s_i \end{bmatrix} \quad \text{for } i = 1, 2,
\end{aligned}$$

which completes the proof. □

Theorem 13. (Dilation) *Let us consider the matrix parameters* $A_i = \begin{bmatrix} a_i & b_i & | & r_i \\ c_s & d_i & | & s_i \end{bmatrix}$ *and* $B_i = \begin{bmatrix} a_i & \frac{b_i}{m^2} & | & \frac{r_i}{m} \\ m^2 c_s & d_i & | & m s_i \end{bmatrix}$, *for $i = 1, 2$. Then, for $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$, we have*

$$\mathcal{A}_{\mathcal{D}_m f, \mathcal{D}_m g}^{A_1,A_2}(\mathbf{t}, \mathbf{w}) = \mathcal{A}_{f,g}^{B_1,B_2}\left(\frac{\mathbf{t}}{m}, \frac{\mathbf{w}}{m}\right), \tag{31}$$

where $\mathcal{D}_m f(\mathbf{t}) = \frac{1}{m} f\left(\frac{\mathbf{t}}{m}\right), \quad m \neq 0.$

Proof. The proof follows by using the procedure of Theorem 5. □

Theorem 14. (Nonlinearity) *Let f and g be two quaternion functions in $L^2(\mathbb{R}^2, \mathbb{H})$. Then,*

$$\mathcal{A}_{f+g}^{A_1,A_2} = \mathcal{A}_{f,f}^{A_1,A_2} + \mathcal{A}_{f,g}^{A_1,A_2} + \mathcal{A}_{g,f}^{A_1,A_2} + \mathcal{A}_{g,g}^{A_1,A_2}. \tag{32}$$

Proof. The proof follows by using the procedure of Theorem 8. □

Next theorem assures that every original quaternion signal can be uniquely reconstructed by the AF-QOLCT with a quaternion constant.

Theorem 15. (Reconstruction formula for AF-QOLCT). *For $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$, where g does not vanish at* 0, *we obtain the following inversion formula of the AF-QOLCT*:

$$f(\mathbf{t}) = \frac{1}{\overline{g(\mathbf{0})}} \int_{\mathbb{R}^2} K_{A_1}^{-i}\left(w_1, \frac{t_1}{2}\right) \mathcal{A}_{f,g}^{A_1,A_2}(\mathbf{t}, \mathbf{w}) K_{A_2}^{-j}\left(w_2, \frac{t_2}{2}\right) \mathrm{d}^2\mathbf{w}. \tag{33}$$

Proof. It is clear from (29) and (18) that

$$f\left(\mathbf{z} + \frac{\mathbf{t}}{2}\right) \overline{g\left(\mathbf{z} - \frac{\mathbf{t}}{2}\right)} = \int_{\mathbb{R}^2} K_{A_1}^{-i}(w_1, z_1) \mathcal{A}_{f,g}^{A_1,A_2}(\mathbf{t}, \mathbf{w}) K_{A_2}^{-j}(w_2, z_2) \mathrm{d}^2\mathbf{w}.$$

On setting $z = \frac{\mathbf{t}}{2}$, we have from the above equation

$$f(\mathbf{t})\overline{g(\mathbf{0})} = \int_{\mathbb{R}^2} K_{A_1}^{-i}\left(w_1, \frac{t_1}{2}\right) \mathcal{A}_{f,g}^{A_1,A_2}(\mathbf{t}, \mathbf{w}) K_{A_2}^{-j}\left(w_2, \frac{t_2}{2}\right) \mathrm{d}^2\mathbf{w},$$

which implies

$$f(\mathbf{t}) = \frac{1}{\overline{g(\mathbf{0})}} \int_{\mathbb{R}^2} K_{A_1}^{-i}\left(w_1, \frac{t_1}{2}\right) \mathcal{A}_{f,g}^{A_1,A_2}(\mathbf{t}, \mathbf{w}) K_{A_2}^{-j}\left(w_2, \frac{t_2}{2}\right) \mathrm{d}^2\mathbf{w}.$$ □

5 Conclusion

Based on the QOLCT and the classical WVD theory, this article first proposes the definition of WVD in the QOLCT domain, namely WVD-QOLCT. Various properties of the WVD-QOLCT, including dilation, shift, and boundedness, are derived in detail. Next, we propose the definition of ambiguity function associated with QOLCT (AF-QOLCT) and establish various properties associated with it viz nonlinearity, dilation and reconstruction formula. The obtained results are new and meaningful. This article is useful for studying the areas of numerical simulations and empirical analysis like [27–30].

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