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Skew Laplacian energy of digraphs

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Abstract In this paper, we consider the Laplacian energy of digraphs. Various approaches for the Laplacian energy of a digraph have been put forward by different authors. We consider the skew Laplacian energy of a digraph as given in Cai et al. (Trans Combin 2:27–37, 2013). We obtain some upper and lower bounds for the skew Laplacian energy which are better than some previous known bounds. We also show every even positive integer is the skew Laplacian energy of some digraphs.

Keywords Laplacian spectra · Skew-Laplacian spectra · Skew-Laplacian energy of a diagraph

Mathematics Subject Classification 05C50 · 05C30

1 Introduction

Let $\mathscr{D}$ be a digraph with n vertices $v_1, v_2, \ldots, v_n$ and m arcs. Let $d_i^+ = d^+(v_i)$, $d_i^- = d^-(v_i)$ and $d_i = d_i^+ + d_i^-$, $i = 1, 2, \ldots, n$ be the outdegree, indegree and degree of the vertices of $\mathscr{D}$, respectively. The out-adjacency matrix $A^+(\mathscr{D}) = (a_{ij})$ of a digraph $\mathscr{D}$ is the $n \times n$ matrix, where $a_{ij} = 1$, if (v_i, v_j) is an arc and $a_{ij} = 0$, otherwise. The in-adjacency matrix $A^-(\mathscr{D}) = (a_{ij})$ of a digraph $\mathscr{D}$ is the $n \times n$ matrix, where $a_{ij} = 1$, if (v_j, v_i) is an arc and $a_{ij} = 0$, otherwise. It is clear that $A^-(\mathscr{D}) = (A^+(\mathscr{D}))^t$.

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The skew adjacency matrix $S(\mathscr{D}) = (s_{ij})$ of a digraph $\mathscr{D}$ is the $n \times n$ matrix, where

$$s_{ij} = \begin{cases} 1, & \text{if there is an arc from } v_i \text{ to } v_j, \\ -1, & \text{if there is an arc from } v_j \text{ to } v_i, \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that $S(\mathscr{D})$ is a skew symmetric matrix, so all its eigenvalues are zero or purely imaginary. The energy of the matrix $S(\mathscr{D})$ was considered in [1], and is defined as

$$E_s(\mathscr{D}) = \sum_{i=1}^{n} |\xi_i|,$$

where $\xi_1, \xi_2, \ldots, \xi_n$ are the eigenvalues of $S(\mathscr{D})$. This energy of a digraph $\mathscr{D}$ is called the skew energy by Adiga et al. [1]. For recent developments in the theory of skew energy, see the survey [17].

Let $D^+(G) = diag(d_1^+, d_2^+, \ldots, d_n^+)$, $D^-(G) = diag(d_1^-, d_2^-, \ldots, d_n^-)$ and $D(G) = diag(d_1, d_2, \ldots, d_n)$ be the diagonal matrix of vertex outdegrees, vertex indegrees and vertex degrees of $\mathscr{D}$ respectively.

Many results have been obtained on the skew spectra and skew spectral radii of oriented graphs [2,4–7,14,21–23].

Recently (in 2013) Cai et al. [3] defined a new type of skew Laplacian matrix $\widetilde{SL}(\mathscr{D})$ of a digraph $\mathscr{D}$ as follows.

Let $D^+(\mathscr{D})$ and $D^-(\mathscr{D})$ respectively be the diagonal matrices of vertex outdegree and vertex indegree and let $A^+(\mathscr{D})$ and $A^-(\mathscr{D})$ respectively be the out-adjacency and in-adjacency matrix of a digraph $\mathscr{D}$. If $A(G)$ is the adjacency matrix of the underlying graph G of the digraph $\mathscr{D}$, then it is clear that $A(G) = A^+(\mathscr{D}) + A^-(\mathscr{D})$ and $S(\mathscr{D}) = A^+(\mathscr{D}) - A^-(\mathscr{D})$, where $S(\mathscr{D})$ is the skew adjacency matrix of $\mathscr{D}$. Therefore, following the definition of Laplacian matrix of a graph, Cai et al. called the matrix

$$\begin{aligned} \widetilde{SL}(\mathscr{D}) &= (D^+(\mathscr{D}) - D^-(\mathscr{D})) - (A^+(\mathscr{D}) - A^-(\mathscr{D})) \\ &= \widetilde{D}(\mathscr{D}) - S(\mathscr{D}), \end{aligned}$$

where $\widetilde{D}(\mathscr{D}) = D^+(\mathscr{D}) - D^-(\mathscr{D})$, as the skew Laplacian matrix of the digraph $\mathscr{D}$. It is clear that the matrix $\widetilde{SL}(\mathscr{D})$ is not symmetric, so its eigenvalues need not be real. However, we have the following observation.

Theorem 1 (i) *$\nu_1, \nu_2, \ldots, \nu_n$ are the eigenvalues of $\widetilde{SL}(\mathscr{D})$, then $\sum_{i=1}^{n} \nu_i = 0$.*
(ii) *0 is an eigenvalue of $\widetilde{SL}(\mathscr{D})$ with multiplicity at least p, where p is the number of components of $\mathscr{D}$ with all ones vector $(1, 1, \ldots, 1)$ as the corresponding eigenvector.*

Following the definition of matrix energy given by Nikifrov [18], Cai et al. [3] defined the skew Laplacian energy of a digraph $\mathscr{D}$, as the sum of the absolute values of the eigenvalues of the matrix $\widetilde{SL}(\mathscr{D})$ and obtained various bounds.

The rest of the paper is organized as follows. In Sect. 2, we obtain the skew Laplacian energy of a star for any orientation and a cycle for some orientations. In Sect. 3, we mention some known bounds for $SLE(\mathscr{D})$. We also obtain some bounds for $SLE(\mathscr{D})$ which are better than already known bounds.

2 Laplacian energy of digraphs

Definition 2.1 Skew Laplacian energy of a digraph. Let $\mathscr{D}$ be a digraph of order n with m arcs and having skew Laplacian eigenvalues $\nu_1, \nu_2, \ldots, \nu_n$. The skew Laplacian energy of $\mathscr{D}$ is denoted by $SLE(\mathscr{D})$ and is defined as

$$SLE(\mathscr{D}) = \sum_{j=1}^{n} |\nu_j|. \tag{2.1}$$

This concept was introduced in 2013 by Cai et al. [3]. The idea of Cai et al. was to conceive a graph energy like quantity for a digraph, that instead of skew adjacency eigenvalues is defined in terms of skew Laplacian eigenvalues and that hopefully would preserve the main features of the original graph energy. The definition of $SLE(\mathscr{D})$ was therefore so chosen that all the properties possessed by graph energy should be preserved.

The topic of energy of graphs and digraphs is an active component of the present research, and various papers have been published in this direction. For the recent papers on the energy of graphs and digraphs and related results, we refer to [9]–[12], [19] and the references therein. For any undefined definition or notation, we refer to [20].

A digraph $\mathscr{D}$ is said to be Eulerian if $d_i^+ = d_i^-$, for all $i = 1, 2, \ldots, n$. Therefore, for an Eulerian digraph $\mathscr{D}$, we always have $\widetilde{D}(\mathscr{D}) = 0$, which gives $\widetilde{SL}(\mathscr{D}) = -S(\mathscr{D})$. Using this, we have the following observation.

Theorem 2.2 *For an Eulerian digraph $\mathscr{D}$, $SLE(\mathscr{D}) = E_s(\mathscr{D})$, where $E_s(\mathscr{D})$ is the skew energy of $\mathscr{D}$.*

As an immediate consequence to Theorem 2.2, we have the following result.

Corollary 2.3 *For a directed cycle C_n, $SLE(C_n) = E_s(C_n)$, where $E_s(\mathscr{D})$ is the skew energy of $\mathscr{D}$.*

We show that every even positive integer is indeed the skew Laplacian energy of some digraph.

Theorem 2.4 *Every even positive integer $2(n-1)$ is the skew Laplacian energy of a directed star of order $n+1$.*

Proof Let $V(K_{1,n}) = \{v_1, v_2, \ldots, v_{n+1}\}$ be the vertex set of $K_{1,n}$. If v_{n+1} is the center of $K_{1,n}$, orient all the edges toward v_{n+1}. Then $S(K_{1,n}) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -1 & -1 & \cdots & -1 & 0 \end{pmatrix}$ and $\widetilde{D}(K_{1,n}) = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & -n \end{pmatrix}$.

Therefore,

$$\widetilde{SL}(K_{1,n}) = \begin{pmatrix} 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \\ 1 & 1 & \cdots & 1 & -n \end{pmatrix}.$$

It is easy to see that the eigenvalues of this matrix are $\{-(n-1), 0, 1^{[n-1]}\}$, and so $SLE(K_{1,n}) = 2(n-1)$. On the other hand, if we orient the edges away from v_{n+1}, then it can be seen that $\widetilde{SL}(K_{1,n}) = \begin{pmatrix} -1 & 0 & \cdots & 0 & 1 \\ 0 & -1 & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 1 \\ -1 & -1 & \cdots & -1 & n \end{pmatrix}$, having eigenvalues $\{(n-1), 0, -1^{[n-1]}\}$, so $SLE(K_{1,n}) = 2(n-1)$. Thus, for a directed star $K_{1,n}$, we have $SLE(K_{1,n}) = 2(n-1)$. □

If all the edges of the star $K_{1,n}$ are oriented away from the center v_{n+1} except k, $1 \le k \le n-1$, edges which are oriented towards the center v_{n+1}, then it can be seen that the skew Laplacian matrix of $K_{1,n}$ is

$$\widetilde{SL}(K_{1,n}) = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & -1 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & -1 \\ 0 & 0 & \cdots & 0 & -1 & \cdots & 0 & 1 \\ \vdots & \vdots & \cdots & 0 & 0 & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & -1 & 1 \\ 1 & 1 & \cdots & 1 & -1 & \cdots & -1 & n-2k \end{pmatrix}.$$

By direct calculation, it can be seen that the skew Laplacian characteristic polynomial of this matrix is $x(x-1)^{k-1}(x+1)^{n-k-1}\left(x^2-(n-2k)x+n-1\right)$ and so its eigenvalues are $\left\{0, 1^{[k-1]}, -1^{[n-k-1]}, \frac{n-2k+\sqrt{(n-2k)^2-4(n-1)}}{2}, \frac{n-2k-\sqrt{(n-2k)^2-4(n-1)}}{2}\right\}$.

Therefore, $SLE(K_{1,n}) = n-2+\sqrt{(n-2k)^2-4(n-1)}$. Thus, using Theorem 2.4, we have $SLE(K_{1,n}) = 2(n-1)$, if all the edges are oriented towards or away from the center, and $SLE(K_{1,n}) = n-2+\sqrt{(n-2k)^2-4(n-1)}$, otherwise, where k, $1 \le k \le n-1$ is the number of edges oriented towards the center, giving the complete description of the skew Laplacian energy of orientations of $K_{1,n}$. It is clear that unlike the skew energy of any orientation of $K_{1,n}$, which is same as the corresponding energy, the skew Laplacian energy of orientations of $K_{1,n}$ is not same as the corresponding Laplacian energy.

Moreover, it is also clear that any two orientations which contain edges directed from and directed to, the center of $K_{1,n}$ are mutually non cospectral digraphs.

3 Bounds for skew Laplacian energy

In this section, we mention some well known bounds for skew Laplacian energy $SLE(\mathscr{D})$, which gives its connection to various graph parameters. We obtain some bounds for $SLE(\mathscr{D})$ which are better than already known bounds.

For a digraph with n vertices, m arcs having vertex outdegrees d_i^+ and vertex indegrees d_i^-, $i = 1, 2, \ldots, n$, let $M = -m + \frac{1}{2}\sum_{i=1}^n \left(d_i^+ - d_i^-\right)^2$ and $M_1 = M + 2m = m + \frac{1}{2}\sum_{i=1}^n \left(d_i^+ - d_i^-\right)^2$. Clearly, $M_1 \ge m$, with equality if and only if $\mathscr{D}$ is Eulerian.

The following bounds are obtained in the basic paper [3] for skew Laplacian energy $SLE(\mathscr{D})$ of a digraph $\mathscr{D}$, which are analogues to the corresponding bounds on Laplacian energy $LE(G)$.

Theorem 3.1 *Let $\mathscr{D}$ be a simple digraph possessing n vertices, m arcs and p components. Assume that d_i^+ and d_i^- respectively are the outdegree and indegree of the vertex v_i, $i = 1, 2, \ldots, n$ and $\nu_1, \nu_2, \ldots, \nu_n$ are the skew Laplacian eigenvalues of $\mathscr{D}$. Then*

$$2\sqrt{|M|} \le SLE(\mathscr{D}) \le \sqrt{2M_1(n-p)}. \tag{3.1}$$

Equality occurs on the left if and only if for each pair of $\nu_{i_1}\nu_{j_1}$ and $\nu_{i_2}\nu_{j_2}$ $(i_1 \ne j_1, i_2 \ne j_2)$, there exists a non-negative real number k such that $\nu_{i_1}\nu_{j_1} = k\nu_{i_2}\nu_{j_2}$; and for each pair of $\nu_{i_1}^2$ and $\nu_{i_2}^2$, there exists a non-negative real number l such that $\nu_{i_1}^2 = l\nu_{i_2}^2$. Equality occurs on the right if and only if $\mathscr{D}$ is 0-regular or for each $v_i \in V(\mathscr{D})$, $d_i^+ = d_i^-$, and the eigenvalues of $\widetilde{SL}(\mathscr{D})$ are $0^{[p]}$, $(ai)^{[\frac{n-p}{2}]}$, $(-ai)^{[\frac{n-p}{2}]}$ $(a > 0)$.

As an immediate consequence to Theorem 3.1, we have the following result.

Corollary 3.2 *Let $\mathscr{D}$ be a simple digraph possessing p components $C_1, C_2, \ldots, C_p$. If $SLE(\mathscr{D}) = \sqrt{2M_1(n-p)}$, then each component C_i is Eulerian with odd number of vertices.*

Since $n - p \le n$, we have the following consequence of Theorem 3.1.

Corollary 3.3 *For any simple digraph $\mathscr{D}$, $SLE(\mathscr{D}) \le \sqrt{2M_1 n}$.*

If $\mathscr{D}$ has no isolated vertices, then $n \le 2m$, and so $\sqrt{2M_1 n} \le 2\sqrt{M_1 m} \le 2M_1$. Thus we have the following observation.

Corollary 3.4 *For any simple digraph $\mathscr{D}$, $SLE(\mathscr{D}) \le 2M_1$.*

We now obtain a Koolen type upper bound (see [15]) for $SLE(\mathscr{D})$.

Theorem 3.5 *Let $\mathscr{D}$ be a simple digraph with n vertices, m arcs and p components. Assume that $t = |\nu_1| \ge |\nu_2| \ge \cdots \ge |\nu_{n-p}| \ge 0$, where $\nu_1, \nu_2, \ldots, \nu_{n-p}, 0^{[p]}$ are the eigenvalues of $\widetilde{SL}(\mathscr{D})$. Then*

$$SLE(\mathscr{D}) \le t + \sqrt{(n-p-1)(2M_1 - t^2)}.$$

Equality occurs if and only if $\mathscr{D}$ is 0-regular or for each $v_i \in V(\mathscr{D})$, $d_i^+ = d_i^-$, and the eigenvalues of $\widetilde{SL}(\mathscr{D})$ are $0^{[p]}$, $(ai)^{[\frac{n-p}{2}]}$, $(-ai)^{[\frac{n-p}{2}]}$ $(a > 0)$.

Proof Let $\widetilde{SL}(\mathscr{D}) = (l_{ij})$. By Schur's triangularization theorem [13], there exists a unitary matrix U such that $U^*\widetilde{SL}(\mathscr{D})U = T$, where $T = (t_{ij})$ is an upper triangular matrix with diagonal entries $t_{ii} = \nu_i$, $i = 1, 2, \ldots, n$. Therefore,

$$\sum_{i,j=1}^n |l_{ij}|^2 = \sum_{i,j=1}^n |t_{ij}|^2 \ge \sum_{i=1}^n |t_{ii}|^2 = \sum_{i=1}^n |\nu_i|^2,$$

that is,

$$\sum_{i=1}^n |\nu_i|^2 \le \sum_{i,j=1}^n |l_{ij}|^2 = \sum_{i,j=1}^n (d_i^+ - d_i^-)^2 + 2m = 2M_1. \tag{3.2}$$

Now, applying Cauchy–Schwarz's inequality to vectors $(|\nu_2|, |\nu_3|, \ldots, |\nu_{n-p}|)$ and $(1, 1, \ldots, 1)$ and using (3.2), we have

$$\begin{aligned} SLE(\mathscr{D}) - |\nu_1| &= \sum_{i=2}^n |\nu_i| = \sum_{i=2}^{n-p} |\nu_i| \le \sqrt{(n-p-1)\sum_{i=2}^{n-p} |\nu_i|^2} \\ &= \sqrt{(n-p-1)\sum_{i=2}^n |\nu_i|^2} \le \sqrt{(n-p-1)(2M_1 - |\nu_1|^2)}. \end{aligned}$$

This gives,

$$SLE(\mathscr{D}) \leq t + \sqrt{(n-p-1)(2M_1 - t^2)}.$$

Equality case can be discussed similarly as in Theorem 3.1. $\square$

The following arithmetic–geometric mean inequality can be found in [16].

Lemma 3.6 *If $a_1, a_2, \ldots, a_n$ are non-negative numbers, then*

$$n\left[\frac{1}{n}\sum_{j=1}^{n} a_j - \left(\prod_{j=1}^{n} a_j\right)^{\frac{1}{n}}\right] \leq n\sum_{j=1}^{n} a_j - \left(\sum_{j=1}^{n}\sqrt{a_j}\right)^2$$
$$\leq n(n-1)\left[\frac{1}{n}\sum_{j=1}^{n} a_j - \left(\prod_{j=1}^{n} a_j\right)^{\frac{1}{n}}\right].$$

Moreover equality occurs if and only if $a_1 = a_2 = \cdots = a_n$.

The following inequality was obtained by Furuichi [8].

Lemma 3.7 *For $a_1, a_2, \ldots, a_n \geq 0$ and $p_1, p_2, \ldots, p_n \geq 0$ such that $\sum_{j=1}^{n} p_i = 1$,*

$$\sum_{j=1}^{n} a_j p_j - \prod_{j=1}^{n} a_j^{p_j} \geq n\lambda\left(\frac{1}{n}\sum_{j=1}^{n} a_j - \prod_{j=1}^{n} a_j^{\frac{1}{n}}\right),$$

where $\lambda = \min\{p_1, p_2, \ldots, p_n\}$. Moreover equality occurs if and only if $a_1 = a_2 = \cdots = a_n$.

For a connected digraph $\mathscr{D}$, let $K = \prod_{j=1}^{n-1} |\nu_j|$, where $|\nu_1| \geq |\nu_2| \geq |\nu_{n-1}| \geq 0$ are the absolute values of the eigenvalues of $\widetilde{SL}(\mathscr{D})$.

The following gives a lower bound for $SLE(\mathscr{D})$ in terms of the number of vertices n and the number K.

Theorem 3.8 *Let $\mathscr{D}$ be a simple connected digraph with $n \geq 3$ vertices and m arcs having skew Laplacian eigenvalues $\nu_1, \nu_2, \ldots, \nu_{n-1}, 0$ with $t = |\nu_1| \geq |\nu_2| \geq \cdots \geq |\nu_{n-1}| \geq 0$. Then*

$$SLE(\mathscr{D}) \geq t + (n-2)K^{\frac{1}{n-1}}\left(\frac{K^{\frac{1}{2(n-1)(n-2)}}}{t^{\frac{1}{2n-4}}} - 1\right), \tag{3.3}$$

with equality if and only if $t = |\nu_1| = |\nu_2| = \cdots = |\nu_{n-1}|$.

Proof Setting $n = n-1$, $a_j = |\nu_j|$, for $j = 1, 2, \ldots, n-1$, $p_1 = \frac{1}{2(n-1)}$, $p_j = \frac{2n-3}{2(n-1)(n-2)}$, for $j = 2, 3, \ldots, n-1$ in Lemma 3.7, we have

$$\frac{|\nu_1|}{2(n-1)} + \frac{2n-3}{2(n-1)(n-2)}\sum_{j=2}^{n-1}|\nu_j| - |\nu_1|^{\frac{1}{2(n-1)}}\prod_{j=2}^{n-1}|\nu_j|^{\frac{2n-3}{2(n-1)(n-2)}}$$
$$\geq \frac{1}{2(n-1)}\sum_{j=1}^{n-1}|\nu_j| - \frac{1}{2}\prod_{j=1}^{n-1}|\nu_j|^{\frac{1}{n-1}},$$

that is,

$$\frac{|\nu_1|}{2(n-1)} + \frac{2n-3}{2(n-1)(n-2)}\left(SLE(\mathscr{D}) - |\nu_1|\right) - |\nu_1|^{\frac{-1}{2(n-2)}}K^{\frac{2n-3}{2(n-1)(n-2)}}$$
$$\geq \frac{1}{2(n-1)}SLE(\mathscr{D}) - \frac{1}{2}K^{\frac{1}{n-1}},$$

this gives,

$$SLE(\mathscr{D}) \geq 2(n-2)\left(\frac{|\nu_1|}{2(n-1)} + \frac{K^{\frac{2n-3}{2(n-1)(n-2)}}}{|\nu_1|^{\frac{1}{2(n-2)}}} - \frac{1}{2}K^{\frac{1}{n-1}}\right),$$

from this the result follows.

Equality occurs in (3.3) if and only if equality occurs in Lemma 3.7, that is if and only if $t = |\nu_1| = |\nu_2| = \cdots = |\nu_{n-1}|$. $\square$

We now obtain the bounds for $SLE(\mathscr{D})$ in terms of the number of vertices n, the numbers K, M and M_1.

Theorem 3.9 *Let $\mathscr{D}$ be a simple connected digraph with $n \geq 3$ vertices and m arcs having skew Laplacian eigenvalues $\nu_1, \nu_2, \ldots, \nu_{n-1}, 0$ with $|\nu_1| \geq |\nu_2| \geq \cdots \geq |\nu_{n-1}| \geq 0$. Then*

$$\sqrt{2|M| + (n-1)(n-2)K^{\frac{2}{n-1}}} \leq SLE(\mathscr{D}) \leq \sqrt{2M_1(n-2) + (n-1)K^{\frac{2}{n-1}}}, \tag{3.4}$$

with equality on the left if and only if for each pair $\nu_{i_1}^2$ and $\nu_{i_2}^2$, there exists a non-negative real number l such that $\nu_{i_1}^2 = l\nu_{i_2}^2$ and the equality on right occurs if and only if $\mathscr{D}$ is 0-regular or for each $v_i \in V(\mathscr{D})$, $d_i^+ = d_i^-$, and the eigenvalues of $\widetilde{SL}(\mathscr{D})$ are $0^{[p]}$, $(ai)^{[\frac{n-p}{2}]}$, $(-ai)^{[\frac{n-p}{2}]}(a > 0)$.

Proof Setting $n = n-1$ and $a_j = |\nu_j|^2$, for $j = 1, 2, \ldots, n-1$ in Lemma 3.6, we have

$$\alpha \leq (n-1)\sum_{j=1}^{n-1}|\nu_j|^2 - \left(\sum_{j=1}^{n-1}|\nu_j|\right)^2 \leq (n-2)\alpha,$$

that is,

$$\alpha \leq (n-1)\sum_{j=1}^{n-1}|\nu_j|^2 - (SLE(\mathscr{D}))^2 \leq (n-2)\alpha, \tag{3.5}$$

where

$$\alpha = (n-1)\left[\frac{1}{n-1}\sum_{j=1}^{n-1}|\nu_j|^2 - \left(\prod_{j=1}^{n-1}|\nu_j|^2\right)^{\frac{1}{n-1}}\right]$$
$$= \sum_{j=1}^{n-1}|\nu_j|^2 - (n-1)\left(\prod_{j=1}^{n-1}|\nu_j|\right)^{\frac{2}{n-1}}$$
$$= \sum_{j=1}^{n-1}|\nu_j|^2 - (n-1)K^{\frac{2}{n-1}}.$$

Using (3.2) and the value of α, we have from the left inequality of (3.5)

$$(SLE(\mathscr{D}))^2 \leq (n-2)\sum_{j=1}^{n-1}|\nu_j|^2 + (n-1)K^{\frac{2}{n-1}},$$

that is,

$$SLE(\mathscr{D}) \leq \sqrt{2M_1(n-2) + (n-1)K^{\frac{2}{n-1}}},$$

this proves the right inequality.

Now, using (7) from [3] and the value of α, we have from the right inequality of (3.5)

$$(SLE(\mathscr{D}))^2 \geq \sum_{j=1}^{n-1}|\nu_j|^2 + (n-1)(n-2)K^{\frac{2}{n-1}},$$

that is,

$$SLE(\mathscr{D}) \geq \sqrt{2|M| + (n-1)(n-2)K^{\frac{2}{n-1}}},$$

this proves the left inequality.

Equality case can be discussed similarly as in Theorem 3.1. □

Remark 3.10 The upper bound given by Theorem 3.9, is better than the upper bound given by Theorem 3.1 for all connected digraphs $\mathscr{D}$. As by arithmetic–geometric mean inequality, we have

$$2M_1 \geq \sum_{j=1}^{n-1}|\nu_j|^2 \geq (n-1)\left(\prod_{j=1}^{n-1}|\nu_j|\right)^{\frac{2}{n-1}} = (n-1)K^{\frac{2}{n-1}},$$

adding $2M_1(n-2)$ on both sides, we obtain

$$2M_1(n-1) \geq 2M_1(n-2) + (n-1)K^{\frac{2}{n-1}},$$

from which the result follows.

Remark 3.11 The lower bound given by Theorem 3.9, is better than the lower bound given by Theorem 3.1 for all connected digraphs $\mathscr{D}$, with $2|M| \leq (n-1)(n-2)K^{\frac{2}{n-1}}$.

4 Conclusion

We conclude this paper with the following problems. These problems have been already considered for different energies associated with the graphs and digraphs. Therefore, it will be interest in the future research to study these problems.

Problem 4.1 Interpret all the coefficients of the characteristic polynomial of $\widetilde{SL}(\mathscr{D})$ in terms of $\mathscr{D}$.

Problem 4.2 Establish the possible relations between the largest and smallest skew Laplacian eigenvalue of a digraph $\mathscr{D}$ with the parameters associated with the digraph.

Problem 4.3 Establish the possible relations between the skew Laplacian spectrum of a digraph $\mathscr{D}$ and the Laplacian spectrum of the corresponding underlying graph $G_{\mathscr{D}}$.

Problem 4.4 For any orientation, give the complete description for the skew Laplacian energy of the cycle C_n.

Problem 4.5 Characterise all the non-Eulerian digraphs $\mathscr{D}$ for which $SLE(\mathscr{D}) = E_s(\mathscr{D})$.

Problem 4.6 If possible, interpret skew Laplacian energy in chemistry and other disciplines.

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