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Original article

New Arctan-generator family of distributions with an example of Frechet distribution: Simulation and analysis to strength of glass and carbon fiber data

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ABSTRACT

Since standard distributions do not fundamentally have an acceptable fit to all types of data sets, it is necessary to construct extensions of standard distributions to increase their capability in data modeling. As a result of this shortage in old ones, we proposed a novel generator based on the trigonometric function (Arctan). We selected the Frechet distribution as the baseline for the generator's applicability. This generator produces the "new Arctan Frechet distribution" (NATFD). The fundamental properties of the proposed distribution have been taken into consideration. Estimating the given distribution's parameters is accomplished using the maximum-likelihood method. A simulation study is carried out to evaluate the superiority of the proposed distribution, and two actual data sets are used.

1. Introduction

In the field of distribution theory, one of the most prevalent concerns is the development of novel methods that can be used to expand the existing families of lifetime distributions. Numerous distributions have been successfully applied and utilized to conduct statistical research in various fields, including actuarial science, biological science, demography, social science, engineering, and medical research.

Various lifetime distributions can be found in the research literature and utilized in data analysis. Classical distributions, however, are not sufficient for accurately characterizing and interpreting real-world events in various settings. As a result, it would appear that the existing traditional statistical distributions require expansion and modification. For more information about generalized distributions, see [1–6]. Numerous endeavors have been undertaken to create new categories of lifetime distributions, expanding various existing distribution families and enhancing the adaptability of novel models. During the past few years, various researchers have contributed to developing new classes of lifetime distributions, which can now be found in the statistical literature. Employing trigonometric functions for building novel statistical distributions is becoming a more prevalent approach.

Furthermore, the utilization of these trigonometric distributions in the interpretation of data demonstrates a greater degree of versatility. When we look at the published works, we discover that many authors used different generators and transformations. For more reading see [7–9].

For instance, Eugene et al. [10], the gamma-G family by Zagrafos and Balakrishnan [11], the transformed transformer(T-X) by Alzaatreh et al. [12], the Weibull-G by Bourguignon et al. [13], Brito et al. [14], exponentiated generalized alpha power family by ElSherpieny et al. (2022), and a new method for developing distributions, exemplified by the Rayleigh distribution, employing SS-transformation using trigonometric functions, was introduced by Kumar et al. [15], as discussed by

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Aijaz et al. [16], Chesneau et al. [17], Terna et al. [18], Mahood and Chesneau [19], Souza et al. [20], Jamal and Chesneau [21], and Aijaz et al. [22]. However, the aforementioned generalization strategies for traditional probability distributions have significant limitations; among them are that incorporating extra characteristics into the probability models improves their adaptability, and these techniques frequently lead to reparameterization obstacles. The quantity of parameters in the model is increasing, rendering it more challenging to determine the characteristics of the model. Furthermore, various expanding techniques lower the tractability of the CDF, making manual computation of statistical features challenging. Different efforts at extension complicated the PDF, resulting in computational issues. Several statistical models recommended in the literature feature many attributes that render them adaptable. Contrary to several authors, creating appropriate estimations employing quantitative resources is challenging. However, it is appealing to establish models with a minimum number of characteristics and a significant level of versatility for describing the data. The primary purpose of this research is to create and examine a unique family of probability distributions that do not require the inclusion of additional parameters yet have a high degree of flexibility for data modeling. This paper aims to introduce the new Arctan-generator distributions, a distinctive family of trigonometric function-based generators. The advantage of this generator is that flexibility is acquired without the insertion of additional parameters, and it is obvious from the data analysis section that this unique generator appears more adaptable than arctan-x by Alkhairy et al. [7].

Let us suppose $F(z;\Theta)$ be the cumulative distribution function (cdf) of a random variable Z and Θ denotes parameter space, then the cumulative distribution function of new Arctan-generator family of distributions is described as.

$$\begin{aligned} F(z;\Theta) &= \frac{4}{\pi}\int_0^{2^{G(z;\Theta)}-1} \frac{1}{1+z^2}dz \\ &= \frac{4}{\pi}\tan^{-1}\left(2^{G(z;\Theta)}-1\right) \quad ; \quad z\in\mathbb{R}, \Theta>0 \end{aligned} \tag{1.1}$$

The related probability density function (pdf) of Eq. (1.1) is stated as

$$f(z;\Theta) = \frac{\ln(16)}{\pi}\frac{2^{G(z;\Theta)}g(z;\Theta)}{1+\left(2^{G(z;\Theta)}-1\right)^2}; \quad z\in\mathbb{R}, \Theta>0 \tag{1.2}$$

where $\frac{d(G(z;\Theta))}{dz} = g(z;\Theta)$

Additionally, other pertinent aging indicators such as the reliability function denoted as $R(z;\Theta)$, the hazard rate function (HRF) denoted as $H(z;\Theta)$, and the reverse hazard rate function denoted as $h(z;\Theta)$ are outlined in their general forms:

$$\begin{aligned} R(z;\Theta) &= 1-F(z;\Theta) = 1-\frac{4}{\pi}\tan^{-1}\left(2^{G(z;\Theta)}-1\right) \\ H(z;\Theta) &= \frac{f(z;\Theta)}{R(z;\Theta)} = \frac{\ln(16)}{\pi} \\ &\times \frac{2^{G(z;\Theta)}g(z;\Theta)}{\left(1+\left(2^{G(z;\Theta)}-1\right)^2\right)\left(1-\frac{4}{\pi}\tan^{-1}\left(2^{G(z;\Theta)}-1\right)\right)} \\ h(z;\Theta) &= \frac{f(z;\Theta)}{F(z;\Theta)} = \frac{2^{G(z;\Theta)}g(z;\Theta)\ln(2)}{\left(1+\left(2^{G(z;\Theta)}-1\right)^2\right)\tan^{-1}\left(2^{G(z;\Theta)}-1\right)} \end{aligned}$$

The quantile function of the formulated distribution function can be obtained by inverting Eq. (1.1) it has several importance in statistics to determine the median, skewness, kurtosis, etc.

$$Q(u;\Theta) = Q_*\left(\frac{\ln\left(+1\tan(\frac{\pi u}{4};\Theta)\right)}{\ln(2)}\right)$$

where $Q_*(u;\Theta)$ denotes the quantile function associated with $G(z;\Theta)$ and $u\in(0,1)$.

2. New Arctan-Frechet distribution and properties

Frechet distribution is a continuous univariate distribution developed by Maurice Frechet (1927). This distribution, often recognized as the type I extreme value distribution, has attracted immense research attention in recent years, especially in extreme value analysis of extreme events. Frechet distribution has many scientific applications, including life testing and water resource management. The cdf of the Frechet distribution is defined as

$$G(z;\alpha,\beta) = e^{-\alpha z^{-\beta}}; \quad z>0, \alpha,\beta>0 \tag{2.1}$$

The associated probability density function is stated as

$$g(z;\alpha,\beta) = \alpha\beta z^{-(\beta+1)}e^{-\alpha z^{-\beta}}; \quad z>0, \alpha,\beta>0 \tag{2.2}$$

The mean and variance of Frechet distribution are stated as

$$\begin{aligned} \mu &= \alpha^{\frac{1}{\beta}}\Gamma\left(1-\frac{1}{\beta}\right) \quad ; \quad \beta>1 \\ \sigma^2 &= \alpha^{\frac{2}{\beta}}\Gamma\left(1-\frac{2}{\beta}\right)-\left(\alpha^{\frac{1}{\beta}}\Gamma\left(1-\frac{1}{\beta}\right)\right)^2 \quad ; \quad \beta>2 \end{aligned}$$

In this par, we utilize the Fréchet distribution as the foundational distribution for a newly devised generator and explore its statistical properties. By employing Eqs. (2.1) within (2.2), we determine the cdf of the new Arctan-Fréchet distribution.

$$F(z;\alpha,\beta) = \frac{4}{\pi}\tan^{-1}\left(2^{e^{-\alpha z^{-\beta}}}-1\right) \quad ; \quad z>0, \alpha,\beta>0 \tag{2.3}$$

The associated pdf is stated as

$$f(z;\alpha,\beta) = \frac{\ln(16)}{\pi}\frac{\alpha\beta z^{-\beta-1}e^{-\alpha z^{-\beta}}2^{e^{-\alpha z^{-\beta}}}}{1+\left(2^{e^{-\alpha z^{-\beta}}}-1\right)^2} \quad ; \quad z>0, \alpha,\beta>0 \tag{2.4}$$

Fig. 1 Emphasizes several possible NATFD cdf patterns with different parameters. Fig. 2 Focuses on NATFD pdf layouts with different parameters.

The Fig. 1 depicts the variability in the CDF of NATFD due to changes in the parameters α and β. This implies that the time it takes for failures to occur can vary depending on these parameters. Without knowing the specific interpretation of α and β in the context of the NATFD model, it is difficult to provide more specific comments on how they influence the CDF shape. However, generally: Higher values of α and β might lead to steeper CDF curves, indicating a higher probability of failures at earlier times. Lower values might result in shallower curves, suggesting a lower probability of failures in the initial stages.

The figure illustrates how the pdf of the NATFD can vary depending on the values of the parameters α and β. This implies that the likelihood of failures occurring at different times is influenced by these parameters. While the specific interpretation of α and β depends on the underlying NATFD model, Fig. 2 are some general observations about how they might affect the pdf shape: Higher α or β: The pdf might peak earlier (shift leftward), suggesting a higher chance of failures at earlier times. Lower α or β: The pdf might peak later (shift rightward), indicating a lower probability of failures in the initial stages. For a more specific interpretation, it would be helpful to know: The definition of α and β within the context of this particular NATFD model. The specific probability distribution used to model the NATFD. The source of the data or the application where the NATFD is being used.

2.1. Useful expansion of probability density function

In this portion, we present the pdf expansion, which can be implemented for computing numerous properties of the defined generator, such as mean, incomplete moments, mean deviations, etc. Using following Taylor's series in Eq. (1.2), we have

$$\frac{1}{1+x^2} = \sum_{p=o}^{\infty}\frac{(-1)^p}{2p+1}x^{2p+1}$$

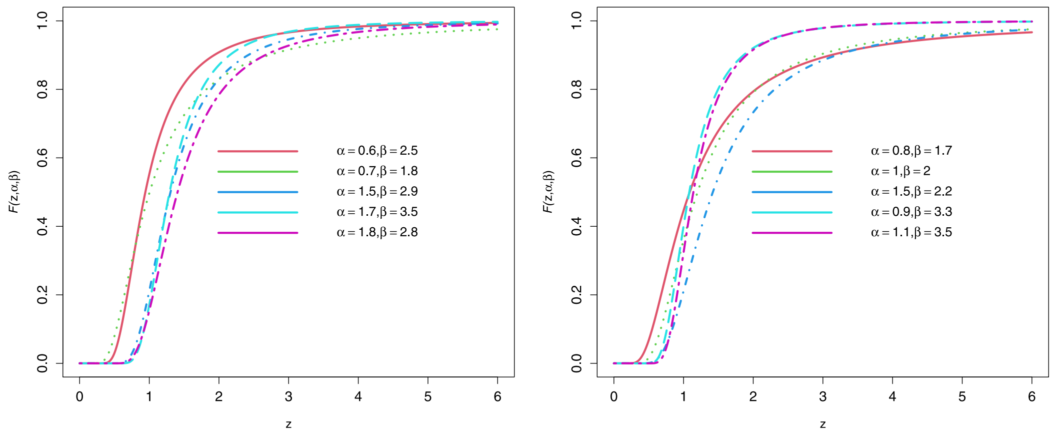

Fig. 1. Emphasizes several possible NATFD cdf patterns with different parameters.

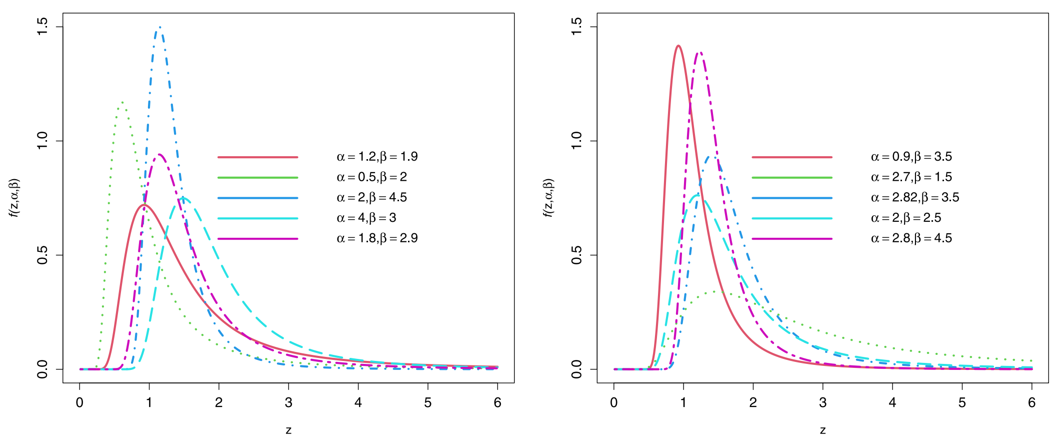

Fig. 2. Emphasizes several possible NATFD pdf patterns with different parameters.

$$f(z;\Theta)=\sum_{p=0}^{\infty}\frac{(-1)^{3p+1}}{2p+1}\frac{\ln(16)}{\pi}2^{G(z;\Theta)}g(z;\Theta)\left(1-2^{G(z;\Theta)}\right)^{2p+1} \tag{2.5}$$

Now, using generalized binomial theorem $(1-u)^a=\sum_{q=0}^{\infty}(-1)^q\binom{a}{q}u^q$ in Eq. (2.5), we have

$$f(z;\Theta)=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(-1)^{3p+q+1}}{2p+1}\frac{\ln(16)}{\pi}\binom{2p+1}{q}2^{(1+q)G(z;\Theta)}g(z;\Theta) \tag{2.6}$$

Again using Taylor's series of $b^x=\sum_{j=0}^{\infty}\frac{(\ln(b))^j}{j!}x^j$ in Eq. (2.6), we have

$$\begin{aligned}f(z;\Theta)&=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{3p+q+1}}{(2p+1)j!}\frac{4(\ln(2))^{j+1}}{\pi}\\&\quad\times\binom{2p+1}{q}(q+1)^j(G(z;\Theta))^jg(z;\Theta)\\&=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{pqj}(G(z;\Theta))^jg(z;\Theta)\end{aligned} \tag{2.7}$$

where

$$\Delta_{pqj}=\frac{(-1)^{3p+q+1}}{(2p+1)j!}\frac{4(\ln(2))^{j+1}}{\pi}\binom{2p+1}{q}(q+1)^j$$

2.2. Moments

Assume Z is a random variable following NATFD. Then the kth moment denoted by μ_k' is defined as

$$\mu_k'=\int_0^{\infty}z^kf(z;\alpha,\beta)dz \tag{2.8}$$

Using Eqs. (2.7) in (2.8), we have

$$\mu_k'=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{pqj}\int_0^{\infty}z^k(G(z;\Theta))^jg(z;\Theta)dz \tag{2.9}$$

Now using Eqs. (2.1) and (2.2) in (2.9), we have

$$\mu_k'=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{pqj}\alpha\beta\int_0^{\infty}z^{k-\beta-1}e^{-\alpha(j+1)z^{-\beta}}dz$$

Making substitution $\alpha(j+1)z^{-\beta}=t$ with $0<t<\infty$, we have

$$\mu_k'=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{pqj}\alpha\left(\alpha(j+1)\right)^{\frac{k}{\beta}-1}\int_0^{\infty}t^{-\frac{k}{\beta}}e^{-t}dt$$

Now solving the integral, we obtain

$$\mu_k'=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{pqj}\alpha\left(\alpha(j+1)\right)^{\frac{k}{\beta}-1}\Gamma\left(1-\frac{k}{\beta}\right)\quad;\quad\beta>k$$

where $\Gamma(n)=\int_0^{\infty}y^{n-1}e^{-y}dy\ ;\ n>0$ denotes a gamma function. Also, the mean exists when $\beta>1$ and variance exists when $\beta>2$ for NATFD. For $k=-1$, we obtain the Harmonic mean, and on substituting $k=1,2,3,4$, we calculate the initial four moments of the specialized distribution. The first moment, referred to as μ_1', serves as its mean, variance, and coefficient of variance (CV), and is defined as:

$$Var=\mu_2=\mu_2'-\mu_1'^2$$

and

$$CV=\frac{\sqrt{\mu_2'-\mu_1'^2}}{\mu_1'}$$

2.3. Moment generating function (MGF)

Assume Z is a random variable following NATFD. Then the MGF is denoted by $M_Z(t)$ is defined as

$$M_Z(t) = E(e^{tz}) = \int_0^\infty e^{tz} f(z;\alpha,\beta)dz$$
$$= \int_0^\infty \left(1 + tz + \frac{(tz)^2}{2!} + \frac{(tz)^3}{3!} + \cdots\right) f(z;\alpha,\beta)dz$$
$$= \sum_{k=0}^\infty \frac{t^k}{k!} E(z^k)$$
$$= \sum_{p=0}^\infty \sum_{q=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \Delta_{pqj} \frac{t^k \alpha\,(\alpha(j+1))^{\frac{k}{\beta}-1}}{k!} \Gamma\left(1 - \frac{k}{\beta}\right)$$

2.4. Incomplete moments

The incomplete moment of order s for a density function is typically expressed as:

$$I_s(v) = \int_0^v z^s f(z;\alpha\beta)dz$$

Using Eq. (2.7), we have

$$I_s(v) = \sum_{p=0}^\infty \sum_{q=0}^\infty \sum_{j=0}^\infty \Delta_{pqj}\alpha\beta \int_0^v z^{s-\beta-1} e^{-\alpha(j+1)z^{-\beta}} dz$$

Making substitution $\alpha(j+1)z^{-\beta} = t$, so that $\alpha(j+1)v^{-\beta} < t < \infty$, we have

$$I_s(v) = \sum_{p=0}^\infty \sum_{q=0}^\infty \sum_{j=0}^\infty \Delta_{pqj}\alpha\,(\alpha(j+1))^{\frac{s}{\beta}-1} \int_{\alpha(j+1)v^{-\beta}}^\infty t^{-\frac{s}{\beta}} e^{-t} dt$$

After solving the integral, we obtain

$$I_s(v) = \sum_{p=0}^\infty \sum_{q=0}^\infty \sum_{j=0}^\infty \Delta_{pqj}\alpha\,(\alpha(j+1))^{\frac{s}{\beta}-1}\, \Gamma\left(\frac{\beta-s}{\beta}, \alpha(j+1)v^{-\beta}\right)$$

where $\Gamma(a,x) = \int_x^\infty u^{a-1}e^{-u}du$ denotes upper incomplete gamma function.

2.5. Quantile function

The quantile function for any distribution can be expressed as follows:

$$Q(u) = Z_u = F^{-1}(u)$$

Here, $Q(u)$ represents the quantile function of $F(z)$ for u in the interval $(0,1)$.

Let us assume

$$F(z) = \frac{4}{\pi}\tan^{-1}\left(2^{e^{-\alpha z^{-\beta}}} - 1\right) = u \tag{2.10}$$

After simplifying Eq. (2.3), we obtain

$$Q(u) = Z_u = \left\{\frac{-1}{\alpha} ln\left[\frac{1}{\ln(2)} ln\left(1 + \tan(\frac{\pi u}{4})\right)\right]\right\}^{-\frac{1}{\beta}}$$

Quantile metrics like Bowley's skewness and Moors kurtosis can be used to assess the fluctuation in skewness and kurtosis. Bowley's skewness, which relies on quartiles, is defined as:

$$\gamma_3 = \frac{Q(0.75) + Q(0.25) - 2Q(0.5)}{Q(0.75) - Q(0.25)}$$

The Moors kurtosis calculated from octiles is defined as:

$$\gamma_4 = \frac{Q(0.37) + Q(0.12) + Q(0.87) - Q(0.62)}{Q(0.75) - Q(0.25)}$$

Table 1 displays different statistical measures of NATFD with various parameter settings. The table's results are derived by applying the NATFD quantile function, which shows that as the parameters

Table 1
The Min., mean, variance, Max., γ_3 and γ_4 for NATFD for distinct parameter combinations.

α	β	Min.	Mean	Variance	Max.	γ_3	γ_4
0.4	2	0.2313	1.0797	1.8310	26.5701	0.2909	1.5676
0.4	2.35	0.2877	1.0244	0.8640	16.3024	0.2651	1.5129
0.4	2.7	0.3381	0.9967	0.5007	11.3529	0.2457	1.4759
0.4	3.05	0.3829	0.9816	0.3283	8.5907	0.2307	1.4493
0.4	3.4	0.4227	0.9731	0.2332	6.8846	0.2188	1.4291
0.4	3.75	0.4581	0.9683	0.1751	5.7501	0.2090	1.4139
0.4	4.1	0.4896	0.9656	0.1367	4.9525	0.2008	1.4016
0.4	4.45	0.5179	0.9642	0.1101	4.3669	0.1937	1.3914
0.4	4.8	0.5433	0.9637	0.0907	3.9219	0.1880	1.3829
0.75	2	0.3167	1.4785	3.4331	36.3827	0.2909	1.5676
0.75	2.35	0.3759	1.3386	1.4752	21.3020	0.2651	1.5128
0.75	2.7	0.4267	1.2579	0.7977	14.3291	0.2457	1.4757
0.75	3.05	0.4705	1.2063	0.4958	10.5569	0.2307	1.4492
0.75	3.4	0.5085	1.1708	0.3376	8.2827	0.2186	1.4292
0.75	3.75	0.5416	1.1450	0.2448	6.7995	0.2088	1.4138
0.75	4.1	0.5708	1.1256	0.1858	5.7731	0.2007	1.4015
0.75	4.45	0.5965	1.1105	0.1460	5.0295	0.1938	1.3914
0.75	4.8	0.6194	1.0985	0.1179	4.4707	0.1880	1.3830
1.1	2	0.3836	1.7905	5.0351	44.0616	0.2909	1.5676
1.1	2.35	0.4424	1.5755	2.0437	25.0726	0.2651	1.5129
1.1	2.7	0.4918	1.4496	1.0593	16.5129	0.2457	1.4758
1.1	3.05	0.5335	1.3677	0.6374	11.9694	0.2308	1.4492
1.1	3.4	0.5691	1.3104	0.4229	9.2703	0.2187	1.4292
1.1	3.75	0.5999	1.2682	0.3003	7.5306	0.2088	1.4138
1.1	4.1	0.6266	1.2358	0.2240	6.3384	0.2007	1.4014
1.1	4.45	0.6501	1.2103	0.1734	5.4816	0.1938	1.3913
1.1	4.8	0.6708	1.1897	0.1382	4.8420	0.1878	1.3833
1.45	2	0.4404	2.0557	6.6372	50.5880	0.2909	1.5676
1.45	2.35	0.4976	1.7720	2.5854	28.2003	0.2650	1.5129
1.45	2.7	0.5447	1.6058	1.2998	18.2919	0.2456	1.4758
1.45	3.05	0.5841	1.4973	0.7640	13.1041	0.2307	1.4491
1.45	3.4	0.6173	1.4213	0.4975	10.0550	0.2186	1.4292
1.45	3.75	0.6457	1.3651	0.3479	8.1064	0.2089	1.4137
1.45	4.1	0.6703	1.3220	0.2563	6.7802	0.2007	1.4014
1.45	4.45	0.6917	1.2879	0.1964	5.8326	0.1938	1.3913
1.45	4.8	0.7106	1.2602	0.1551	5.1288	0.1879	1.3831
1.8	2	0.4907	2.2904	8.2393	56.3638	0.2909	1.5676
1.8	2.35	0.5456	1.9428	3.1077	30.9181	0.2651	1.5129
1.8	2.7	0.5901	1.7397	1.5256	19.8169	0.2457	1.4758
1.8	3.05	0.6270	1.6074	0.8804	14.0668	0.2307	1.4492
1.8	3.4	0.6578	1.5146	0.5650	10.7152	0.2187	1.4293
1.8	3.75	0.6841	1.4461	0.3905	8.5875	0.2089	1.4137
1.8	4.1	0.7066	1.3936	0.2848	7.1474	0.2007	1.4015
1.8	4.45	0.7262	1.3520	0.2164	6.1230	0.1938	1.3914
1.8	4.8	0.7433	1.3183	0.1697	5.3652	0.1880	1.3832
2.15	2	0.5363	2.5032	9.8414	61.6003	0.2909	1.5677
2.15	2.35	0.5884	2.0954	3.6151	33.3464	0.2651	1.5129
2.15	2.7	0.6303	1.8580	1.7402	21.1649	0.2457	1.4757
2.15	3.05	0.6646	1.7038	0.9892	14.9107	0.2306	1.4492
2.15	3.4	0.6931	1.5959	0.6273	11.2900	0.2186	1.4292
2.15	3.75	0.7172	1.5163	0.4293	9.0042	0.2090	1.4138
2.15	4.1	0.7379	1.4553	0.3106	7.4639	0.2007	1.4015
2.15	4.45	0.7557	1.4071	0.2344	6.3725	0.1938	1.3915
2.15	4.8	0.7713	1.3680	0.1828	5.5675	0.1879	1.3831
2.5	2	0.5783	2.6993	11.4435	66.4254	0.2909	1.5677
2.5	2.35	0.6274	2.2343	4.1102	35.5568	0.2651	1.5128
2.5	2.7	0.6665	1.9648	1.9459	22.3809	0.2457	1.4757
2.5	3.05	0.6982	1.7901	1.0920	15.6666	0.2307	1.4491
2.5	3.4	0.7246	1.6682	0.6854	11.8021	0.2187	1.4293
2.5	3.75	0.7467	1.5785	0.4652	9.3737	0.2088	1.4137
2.5	4.1	0.7655	1.5098	0.3343	7.7436	0.2006	1.4014
2.5	4.45	0.7818	1.4556	0.2508	6.5921	0.1938	1.3914
2.5	4.8	0.7960	1.4117	0.1946	5.7452	0.1879	1.3831
2.85	2	0.6174	2.8820	13.0456	70.9229	0.2909	1.5676
2.85	2.35	0.6634	2.3624	4.5951	37.5956	0.2650	1.5129
2.85	2.7	0.6996	2.0625	2.1443	23.4938	0.2456	1.4757
2.85	3.05	0.7289	1.8687	1.1900	16.3542	0.2306	1.4491
2.85	3.4	0.7530	1.7338	0.7404	12.2659	0.2186	1.4293
2.85	3.75	0.7733	1.6347	0.4989	9.7070	0.2089	1.4138
2.85	4.1	0.7904	1.5589	0.3564	7.9951	0.2007	1.4014
2.85	4.45	0.8052	1.4991	0.2660	6.7891	0.1937	1.3914
2.85	4.8	0.8180	1.4507	0.2055	5.9042	0.1879	1.3831

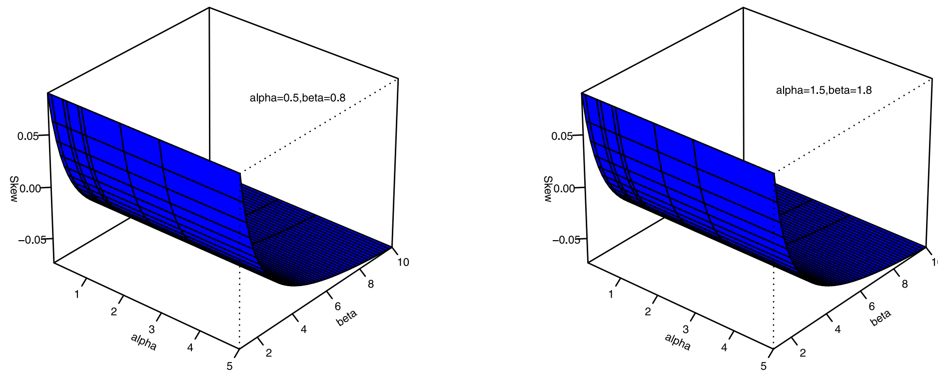

Fig. 3. Bowley's skewness for different parameter choices.

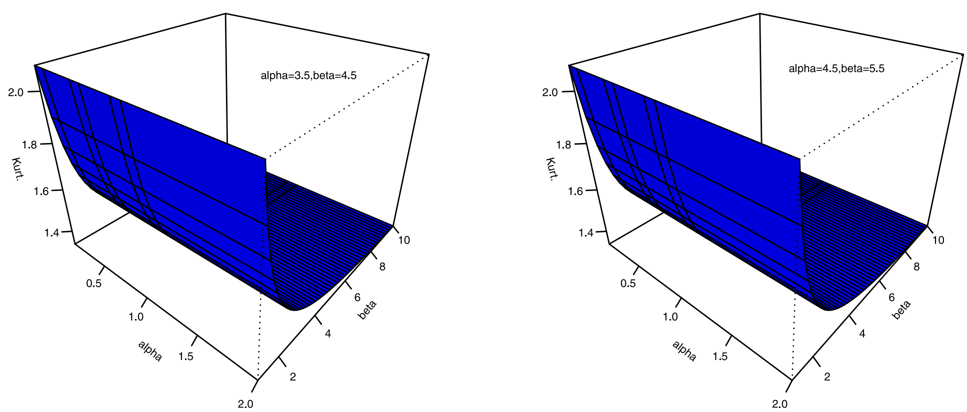

Fig. 4. Moors kurtosis for different parameter choices.

are raised, the Minimum (Min.), mean, variance, Maximum (Max.), values, skewness, and kurtosis improve. As a result, the NATFD is a versatile model that can handle bigger data sets. Fig. 3 displays Bowley's skewness for different parameter choices, and Fig. 4 displays Moors kurtosis for different parameter choices. For more information, see [23–27].

3. Renyi entropy

A random variable's entropy measures the fluctuation of unpredictability. The Renyi entropy is stated as

$$R_\delta = \frac{1}{1-\delta}\ln\left\{\int_0^\infty f^\delta(z;\Theta)dz\right\} \tag{3.1}$$

where $\delta > 0$ and $\delta \neq 1$. Using Eqs. (1.2) in (3.1), we obtain

$$R_\delta = \frac{1}{1-\delta}\ln\left\{\int_0^\infty \left[\frac{\ln(16)}{\pi}\frac{g(z;\Theta)2^{G(z;\Theta)}}{1+\left(2^{G(z;\Theta)}-1\right)^2}\right]^\delta dz\right\} \tag{3.2}$$

Now, using generalized binomial theorem $(1+u)^{-a} = \sum_{j=0}^{\infty}(-1)^j\binom{a+j-1}{j}u^j$; $a > 0, |u| < 1$ in Eq. (3.2), we have

$$R_\delta = \frac{1}{1-\delta}\ln\left\{\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(-1)^{p+q}\binom{\delta+p-1}{p}\binom{2p}{q}\left(\frac{\ln(16)}{\pi}\right)^\delta \times \int_0^\infty (g(z;\Theta))^\delta 2^{(q+\delta)G(z;\Theta)}dz\right\} \tag{3.3}$$

Again using Taylor's series of $b^x = \sum_{j=0}^{\infty}\frac{(\ln(b))^j}{j!}x^j$ in Eq. (3.3), we have

$$R_\delta = \frac{1}{1-\delta}\ln\left\{\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{j=0}^{\infty}\Psi_{pqj}\int_0^\infty (g(z;\Theta))^\delta (G(z;\Theta))^j dz\right\} \tag{3.4}$$

where

$$\Psi_{pqj} = (-1)^{p+q}\binom{\delta+p-1}{p}\binom{2p}{q}\left(\frac{\ln(16)}{\pi}\right)^\delta\frac{(\ln(2))^j}{j!}(\delta+q)^j$$

Substituting Eqs. (2.1) and (2.2) in Eq. (3.4), we have

$$R_\delta = \frac{1}{1-\delta}\ln\left\{\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{j=0}^{\infty}\Psi_{pqj}(\alpha\beta)^\delta\int_0^\infty z^{-\delta(\beta+1)}e^{-\alpha(j+\delta)z^{-\beta}}dz\right\}$$

Making substitution $\alpha(j+\delta)z^{-\beta} = t$ and after some simplification we obtain the desired result as

$$R_\delta = \frac{1}{1-\delta}\ln\left\{\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{j=0}^{\infty}\frac{\Psi_{pqj}\alpha(\alpha\beta)^{\delta-1}}{(\alpha(\delta+j))^{\frac{(\beta+1)(\delta-1)+\beta}{\beta}}}\Gamma\left(\frac{(\beta+1)(\delta-1)+\beta}{\beta}\right)\right\}$$

where $\Gamma(.)$ denotes the gamma function.

4. Aging indicators

This section provides distinct aging indicators, which are defined as follows Let us assume that Z is a random variable following NATFD. Then the reliability function of a random variable is stated as

$$R(z;\alpha,\beta) = 1 - \frac{4}{\pi}\tan^{-1}\left(2^{e^{-\alpha z^{-\beta}}}-1\right) \tag{4.1}$$

The HRF is defined as

$$H(z;\alpha,\beta) = \frac{f(z;\alpha,\beta)}{\bar{F}(z;\alpha,\beta)} \tag{4.2}$$

Now using Eqs. (2.2) and (4.1) in Eq. (4.2), we have

$$H(z;\alpha,\beta) = \frac{\ln(16)\alpha\beta z^{-\beta-1}2^{e^{-\alpha z^{-\beta}}}e^{-\alpha z^{-\beta}}}{\left(\pi - 4\tan^{-1}\left(2^{e^{-\alpha z^{-\beta}}}-1\right)\right)\left(1+\left(2^{e^{-\alpha z^{-\beta}}}-1\right)^2\right)}$$

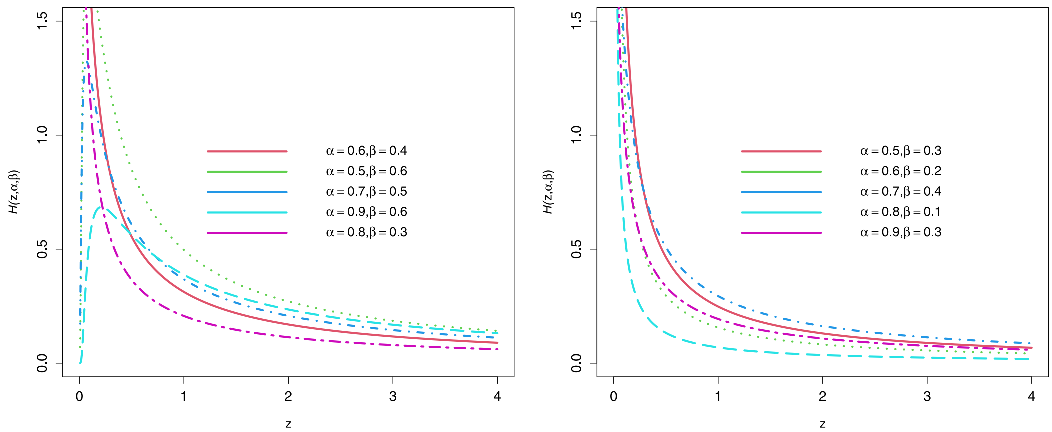

Fig. 5. Emphasizes some of the probable layouts of HRF of NATFD for various choice of parameters.

Fig. 5 discussed emphasizes some of the probable layouts of HRF of NATFD for various choice of parameters. The reverse hazard rate function is defined as

$$r(z;\alpha,\beta)=\frac{f(z;\alpha,\beta)}{F(z;\alpha,\beta)} \tag{4.3}$$

using Eqs. (2.1) and (2.2) in Eq. (4.3), we have

$$r(z;\alpha,\beta)=\frac{\ln(2)\alpha\beta z^{-\beta-1}2^{e^{-\alpha z^{-\beta}}}e^{-\alpha z^{-\beta}}}{\tan^{-1}\left(2^{e^{-\alpha z^{-\beta}}}-1\right)\left(1+\left(2^{e^{-\alpha z^{-\beta}}}-1\right)^{2}\right)}$$

The mean residual lifespan is the component's estimated residual life or average completion time after a specific duration z. Dependability studies rely on it.

It is possible to derive the mean residual function of the random variable z as

$$m(z;\alpha,\beta)=\frac{1}{\bar{F}(z;\alpha,\beta)}\int_{z}^{\infty}tf(t;\alpha,\beta)dt-z \tag{4.4}$$

Using Eqs. (2.7) and (4.1) in Eq. (4.4), we have

$$m(z;\alpha,\beta)=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{pqj}\frac{\alpha\beta\pi}{\pi-4\tan^{-1}\left(2^{e^{-\alpha z^{-\beta}}}-1\right)}\times\int_{z}^{\infty}t^{-\beta}e^{-\alpha(j+1)t^{-\beta}}dt-z \tag{4.5}$$

Making substitution $\alpha(j+1)t^{-\beta}=x$, so that $0<x<\alpha(j+1)z^{-\beta}$, we have

$$m(z;\alpha,\beta)=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{j=0}^{\infty}\frac{\Delta_{pqj}\alpha\pi}{\pi-4\tan^{-1}\left(2^{e^{-\alpha z^{-\beta}}}-1\right)}(\alpha(j+1))^{\frac{1}{\beta}-1}\times\int_{0}^{\alpha(j+1)z^{-\beta}}x^{-\frac{1}{\beta}}e^{-x}dx-z \tag{4.6}$$

After some manipulation, we obtain the desired result as

$$m(z;\alpha,\beta)=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{j=0}^{\infty}\frac{\Delta_{pqj}\alpha\pi}{\pi-4\tan^{-1}\left(2^{e^{-\alpha z^{-\beta}}}-1\right)}(\alpha(j+1))^{\frac{1}{\beta}-1}\times\gamma\left(\frac{\beta-1}{\beta},\alpha(j+1)z^{-\beta}\right)-z$$

where $\gamma(a,x)=\int_0^x u^{a-1}e^{-u}du$ denotes lower incomplete gamma function.

5. Mean deviations about mean and median

Let us consider Z follows NATFD with mean μ. Then the standard deviation from the average is described as in Eq. (5.1)

$$D(\mu)=E\left(|Z-\mu|\right)=2\mu F(\mu)-2\int_{0}^{\mu}zf(z)dz \tag{5.1}$$

Now

$$\int_{0}^{\mu}zf(z)dz=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{pqj}\alpha\beta\int_{0}^{\mu}z^{-\beta}e^{-\alpha(j+1)z^{-\beta}}dz$$

Making substitution $\alpha(j+1)z^{-\beta}=t$, so that $\alpha(j+1)\mu^{-\beta}<t<\infty$, we have

$$\int_{0}^{\mu}zf(z)dz=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{pqj}\alpha\left(\alpha(j+1)\right)^{\frac{1}{\beta}-1}\int_{\alpha(j+1)\mu^{-\beta}}^{\infty}t^{-\frac{1}{\beta}}e^{-t}dt$$

After manipulating the integral, we arrive

$$\int_{0}^{\mu}zf(z)dz=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{pqj}\alpha\left(\alpha(j+1)\right)^{\frac{1}{\beta}-1}\Gamma\left(\frac{\beta-1}{\beta},\alpha(j+1)\mu^{-\beta}\right) \tag{5.2}$$

Using Eqs. (2.1) and (5.2) in Eq. (5.1), we get

$$D(\mu)=\frac{8\mu}{\pi}\tan^{-1}\left(2^{e^{-\alpha\mu^{-\beta}}}-1\right)-2\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{pqj}\alpha\left(\alpha(j+1)\right)^{\frac{1}{\beta}-1}\Gamma\left(\frac{\beta-1}{\beta},\alpha(j+1)\mu^{-\beta}\right)$$

Similarly, if a random variable Z follows NATFD, then the mean deviation from the median is described as

$$D(M)=E\left(|Z-M|\right)=\mu-2\int_{0}^{M}zf(z)dz \tag{5.3}$$

Now

$$\int_{0}^{M}zf(z)dz=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{pqj}\alpha\beta\int_{0}^{M}z^{-\beta}e^{-\alpha(j+1)z^{-\beta}}dz$$

Making substitution $\alpha(j+1)z^{-\beta}=t$, so that $\alpha(j+1)M^{-\beta}<t<\infty$, we have

$$\int_{0}^{M}zf(z)dz=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{pqj}\alpha\left(\alpha(j+1)\right)^{\frac{1}{\beta}-1}\int_{\alpha(j+1)M^{-\beta}}^{\infty}t^{-\frac{1}{\beta}}e^{-t}dt$$

After manipulating the integral, we arrive

$$\int_{0}^{M}zf(z)dz=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{pqj}\alpha\left(\alpha(j+1)\right)^{\frac{1}{\beta}-1}\Gamma\left(\frac{\beta-1}{\beta},\alpha(j+1)M^{-\beta}\right) \tag{5.4}$$

Using Eqs. (5.4) in Eq. (5.3), we get

$$D(M)=\mu-2\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{pqj}\alpha\left(\alpha(j+1)\right)^{\frac{1}{\beta}-1}\Gamma\left(\frac{\beta-1}{\beta},\alpha(j+1)M^{-\beta}\right)$$

6. Order statistics

Let us consider $z_1,z_2,z_3,\dots,z_n$ is the sample under investigating n that follows NATFD. After this is done, the probability density function of the statistics on the kth order may be written as below :

$$f_Z(k) = \frac{n!}{(k-1)!(n-1)!} f(z)\,[F(z)]^{k-1}\,[1-F(z)]^{n-k} \tag{6.1}$$

Using Eqs. (2.1) and (2.2) in Eq. (6.1), we have

$$f_Z(k) = \frac{n!}{(k-1)!(n-1)!}\,\frac{\ln(16)}{\pi}\,\frac{\alpha\beta z^{-\beta-1}e^{-\alpha z^{-\beta}}2^{e^{-\alpha z^{-\beta}}}}{1+\left(2^{e^{-\alpha z^{-\beta}}}-1\right)^2} \times \left[\frac{4}{\pi}\tan^{-1}\left(2^{e^{-\alpha z^{-\beta}}}-1\right)\right]^{k-1}\left[1-\frac{4}{\pi}\tan^{-1}\left(2^{e^{-\alpha z^{-\beta}}}-1\right)\right]^{n-k}$$

The pdf of the first order Z_1 and nth order Z_n statistics of NATFD are respectively given as

$$f_Z(1) = \frac{n\ln(16)}{\pi}\,\frac{\alpha\beta z^{-\beta-1}e^{-\alpha z^{-\beta}}2^{e^{-\alpha z^{-\beta}}}}{1+\left(2^{e^{-\alpha z^{-\beta}}}-1\right)^2}\left[1-\frac{4}{\pi}\tan^{-1}\left(2^{e^{-\alpha z^{-\beta}}}-1\right)\right]^{n-1}$$

$$f_Z(n) = \frac{n\ln(16)}{\pi}\,\frac{\alpha\beta z^{-\beta-1}e^{-\alpha z^{-\beta}}2^{e^{-\alpha z^{-\beta}}}}{1+\left(2^{e^{-\alpha z^{-\beta}}}-1\right)^2}\left[\frac{4}{\pi}\tan^{-1}\left(2^{e^{-\alpha z^{-\beta}}}-1\right)\right]^{n-1}$$

7. Maximum likelihood estimation

Let the random samples $z_1, z_2, z_3, \ldots, z_n$ are drawn from NATFD. The likelihood function of n observations is given as

$$L = \prod_{i=1}^{n}\frac{\ln(16)}{\pi}\,\frac{\alpha\beta z_i^{-\beta-1}e^{-\alpha z_i^{-\beta}}2^{e^{-\alpha z_i^{-\beta}}}}{1+\left(2^{e^{-\alpha z_i^{-\beta}}}-1\right)^2} \tag{7.1}$$

The ln-likelihood of Eq. (7.1) is

$$l = n\ln\left(\frac{16}{\pi}\right)+n\ln(\alpha)+n\ln(\beta)-(\beta+1)\sum_{i=1}^{n}\ln(z_i)+\ln(2)\sum_{i=1}^{n}e^{-\alpha z_i^{-\beta}} - \sum_{i=1}^{n} ln\left(1+\left(2^{e^{-\alpha z_i^{-\beta}}}-1\right)^2\right)-\alpha\sum_{i=1}^{n}z_i^{-\beta} \tag{7.2}$$

Differentiate equation (7.2) partially with respect α and β, we have

$$\frac{\partial l}{\partial\alpha} = \frac{n}{\alpha}-\ln(2)\sum_{i=1}^{n}z_i^{-\beta}e^{-\alpha z_i^{-\beta}}+2\sum_{i=1}^{n}\frac{2^{e^{-\alpha z_i^{-\beta}}}\left(2^{e^{-\alpha z_i^{-\beta}}}-1\right)e^{-\alpha z_i^{-\beta}}z_i^{-\beta}}{1+\left(2^{e^{-\alpha z_i^{-\beta}}}-1\right)^2}-\sum_{i=1}^{n}z_i^{-\beta} \tag{7.3}$$

$$\frac{\partial l}{\partial\beta} = \frac{n}{\beta}-\sum_{i=1}^{n}\ln(z_i)+\alpha\ln(2)\sum_{i=1}^{n}z_i^{-\beta}\ln(z_i)e^{-\alpha z_i^{-\beta}} - 2\alpha\sum_{i=1}^{n}\frac{2^{e^{-\alpha z_i^{-\beta}}}\left(2^{e^{-\alpha z_i^{-\beta}}}-1\right)e^{-\alpha z_i^{-\beta}}z_i^{-\beta}\ln(z_i)}{1+\left(2^{e^{-\alpha z_i^{-\beta}}}-1\right)^2}+\alpha\sum_{i=1}^{n}z_i^{-\beta}\ln(z_i) \tag{7.4}$$

Although we could not acquire a compact form of Eqs. (7.3) and (7.4), we employed iteration approaches such as those of Newton–Raphson and others to address these. An information array is required for interval estimation and hypothesis testing on model parameters. The MLE of the parameters denoted as $\hat{\zeta}(\hat{\alpha},\hat{\beta})$ of $\zeta(\alpha,\beta)$ can be derived by using the above methods.

8. Simulation analysis

Simulation experiments were employed to study the performance of the NATFD parameters. Data sets were produced utilizing the following equation with a replication number of N = 500; a random sample of size n = 20, 50, 150, 250, 350, and 450 was then selected.

$$Z_u = \left\{\frac{-1}{\alpha} ln\left[\frac{1}{\ln(2)} ln\left(1+\tan\left(\frac{\pi u}{4}\right)\right)\right]\right\}^{-\frac{1}{\beta}}$$

Table 2
Bias, variance and their corresponding MSE's at $\alpha = 0.5$.

β	n	Parameters	Estimates	Bias	Variance	MSE
0.4	20	α	0.50352	0.00352	0.02252	0.02251
		β	0.43173	0.03173	0.00728	0.00828
	50	α	0.50189	0.00189	0.00851	0.00851
		β	0.40967	0.00967	0.00217	0.00226
	150	α	0.50011	0.00011	0.00281	0.00281
		β	0.40408	0.00408	0.00075	0.00076
	250	α	0.50027	0.00027	0.00162	0.00162
		β	0.40254	0.00254	0.00041	0.00041
	350	α	0.49948	−0.00052	0.00113	0.00113
		β	0.40141	0.00141	0.00029	0.00029
	450	α	0.49982	−0.00018	0.00086	0.00086
		β	0.40097	0.00097	0.00023	0.00023
0.9	20	α	0.50356	0.00356	0.02247	0.02246
		β	0.97186	0.07186	0.03567	0.04080
	50	α	0.50189	0.00189	0.00851	0.00851
		β	0.92173	0.02173	0.01097	0.01143
	150	α	0.50012	0.00012	0.00281	0.00280
		β	0.90914	0.00914	0.00378	0.00386
	250	α	0.50027	0.00027	0.00162	0.00162
		β	0.90568	0.00568	0.00206	0.00209
	350	α	0.49947	−0.00053	0.00113	0.00113
		β	0.90310	0.00310	0.00148	0.00149
	450	α	0.49988	−0.00012	0.00086	0.00086
		β	0.90201	0.00201	0.00115	0.00115

Table 3
Bias, variance and their corresponding MSE's at $\alpha = 1.2$.

β	n	Parameters	Estimates	Bias	Variance	MSE
1.2	20	α	1.28038	0.08038	0.10027	0.10663
		β	1.29746	0.09746	0.05988	0.06931
	50	α	1.22211	0.02211	0.03214	0.03260
		β	1.22898	0.02898	0.01950	0.02032
	150	α	1.20837	0.00837	0.01017	0.01023
		β	1.21218	0.01218	0.00672	0.00686
	250	α	1.20578	0.00578	0.00579	0.00582
		β	1.20757	0.00757	0.00366	0.00372
	350	α	1.20124	0.00124	0.00392	0.00392
		β	1.20414	0.00414	0.00263	0.00265
	450	α	1.20133	0.00133	0.00324	0.00324
		β	1.20268	0.00268	0.00204	0.00205
3	20	α	1.28038	0.08038	0.10027	0.10663
		β	3.24364	0.24364	0.37421	0.43320
	50	α	1.22211	0.02211	0.03214	0.03260
		β	3.07244	0.07244	0.12186	0.12699
	150	α	1.20837	0.00837	0.01017	0.01023
		β	3.03045	0.03045	0.04200	0.04289
	250	α	1.20578	0.00578	0.00579	0.00582
		β	3.01893	0.01893	0.02289	0.02322
	350	α	1.20124	0.00124	0.00392	0.00392
		β	3.01034	0.01034	0.01645	0.01654
	450	α	1.20133	0.00133	0.00324	0.00324
		β	3.00670	0.00670	0.01276	0.01279

Three separate cases were simulated employing three different sets of parameter values. The MLE of the actual parameters, comprising the bias and root mean square error (RMSE), was computed. Tables 2–5 show that the means square error (MSE) diminishes as the sample size increases for all of the specified parameter values. Furthermore, the estimations' bias is nearer to the actual parameter values. Hence, as the sample size grows, estimates approach parameter values.

Table 4
Bias, variance and their corresponding MSE's at $\alpha = 2.5$.

β	n	Parameters	Estimates	Bias	Variance	MSE
1.5	20	α	2.86044	0.36044	0.75973	0.88888
		β	1.62150	0.12150	0.09319	0.10786
	50	α	2.59776	0.09776	0.18625	0.19562
		β	1.53622	0.03622	0.03047	0.03175
	150	α	2.53861	0.03861	0.05746	0.05890
		β	1.51523	0.01523	0.01050	0.01072
	250	α	2.52485	0.02485	0.03135	0.03194
		β	1.50947	0.00947	0.00572	0.00581
	350	α	2.50971	0.00971	0.02120	0.02128
		β	1.50517	0.00517	0.00411	0.00414
	450	α	2.50770	0.00770	0.01830	0.01834
		β	1.50335	0.00335	0.00319	0.00320
3	20	α	2.86334	0.36334	0.78915	0.92037
		β	3.24361	0.24361	0.37415	0.43312
	50	α	2.59775	0.09775	0.18625	0.19562
		β	3.07244	0.07244	0.12186	0.12699
	150	α	2.53861	0.03861	0.05746	0.05890
		β	3.03045	0.03045	0.04200	0.04289
	250	α	2.52484	0.02484	0.03135	0.03194
		β	3.01893	0.01893	0.02289	0.02322
	350	α	2.50970	0.00970	0.02120	0.02128
		β	3.01034	0.01034	0.01645	0.01654
	450	α	2.50770	0.00770	0.01830	0.01834
		β	3.00670	0.00670	0.01276	0.01279

Table 5
Bias, variance and their corresponding MSE's at $\alpha = 2.5$.

β	n	Parameters	Estimates	Bias	Variance	MSE
2	20	α	4.82303	0.82303	3.11817	3.79243
		β	2.15947	0.15947	0.15996	0.18523
	50	α	4.22399	0.22399	0.70442	0.75389
		β	2.04788	0.04788	0.05379	0.05603
	150	α	4.08906	0.08906	0.21711	0.22482
		β	2.02029	0.02029	0.01867	0.01906
	250	α	4.05584	0.05584	0.11500	0.11800
		β	2.01262	0.01262	0.01017	0.01032
	350	α	4.02493	0.02493	0.07847	0.07902
		β	2.00689	0.00689	0.00731	0.00735
	450	α	4.01897	0.01897	0.06686	0.06715
		β	2.00447	0.00447	0.00567	0.00569
5	20	α	4.85015	0.85015	3.55002	4.26921
		β	5.40480	0.40480	1.02971	1.19255
	50	α	4.22705	0.22705	0.72394	0.77477
		β	5.12070	0.12070	0.33850	0.35273
	150	α	4.08909	0.08909	0.21720	0.22492
		β	5.05072	0.05072	0.11667	0.11913
	250	α	4.05582	0.05582	0.11501	0.11801
		β	5.03152	0.03152	0.06358	0.06451
	350	α	4.02490	0.02490	0.07847	0.07901
		β	5.01720	0.01720	0.04570	0.04595
	450	α	4.01895	0.01895	0.06686	0.06715
		β	5.01114	0.01114	0.03545	0.03554

9. Data analysis

In this portion, we employ real-world data to demonstrate that NATFD is superior to numerous well-known distributions, including the arctan Frechet distribution (ATFD) by Alkhairy et al. [7], Weibull distribution (WD), Frechet distribution (FD), exponentiated exponential distribution (EED), Burr distribution (BD), gamma distribution (GD), inverse Gompertz distribution (IGD), exponential distribution (ED), Gumbel distribution (GBD), Rayleigh distribution (RD) and inverse Rayleigh distribution (IRD).

Table 6
Values of the ML estimates and (standard error in parenthesis) for data set I.

Model	$\hat{\alpha}$	$\hat{\beta}$
NATFD	4.1219 (0.5198)	4.2423 (0.3215)
ATFD	4.5703 (0.53044)	3.9345 (0.30453)
WD	0.1978 (0.03360)	2.6316 (0.16338)
FD	4.3120 (0.54149)	4.3729 (0.32778)
EED	2.8430 (0.23714)	59.139 (19.549)
IGD	0.01507 (0.00546)	5.7472 (0.42518)
BD	19.944 (6.26829)	0.10822 (0.03550)
GD	11.2496 (1.56791)	6.7857 (0.96715)
ED	0.60319 (0.06031)	....
RD	0.32218 (0.03221)	....
GBD	0.80912 (0.06752)	....
IRD	2.20216 (0.22021)	....

The AIC, CAIC, BIC, HQIC, Kolmogorov–Smirnov test (K.S), Cramer-Van Mises criteria (W*), and Anderson–Darling test (A*) are used to evaluate the distribution's adaptability. Better distributions have lower AIC, CAIC, BIC,HQIC, K.S, W*, and A* values.

Data set I: A data set from a study on breaking stress of 100 carbon fibers (in Gba) is considered, which has been studied by Nichols and Padgett [28]. The lifetimes are times until the break of the fibers.

0.92, 0.928, 0.997, 0.9971, 1.061, 1.117, 1.162, 1.183, 1.187, 1.192, 1.196, 1.213, 1.215, 1.2199, 1.22, 1.224, 1.225, 1.228, 1.237, 1.24, 1.244, 1.259, 1.261, 1.263, 1.276, 1.31, 1.321, 1.329, 1.331, 1.337, 1.351, 1.359, 1.388, 1.408, 1.449, 1.4497, 1.45, 1.459, 1.471, 1.475, 1.477, 1.48, 1.489, 1.501, 1.507, 1.515, 1.53, 1.5304, 1.533, 1.544, 1.5443, 1.552, 1.556, 1.562, 1.566, 1.585, 1.586, 1.599, 1.602, 1.614, 1.616, 1.617, 1.628, 1.684, 1.711, 1.718, 1.733, 1.738, 1.743, 1.759, 1.777, 1.794, 1.799, 1.806, 1.814, 1.816, 1.828, 1.83, 1.884, 1.892, 1.944, 1.972, 1.984, 1.987, 2.02, 2.0304, 2.029, 2.035, 2.037, 2.043, 2.046, 2.059, 2.111, 2.165, 2.686, 2.778, 2.972, 3.504, 3.863, 5.306. Table 5 shows values of the ML estimates and (standard error in parenthesis) for data set I. Table 6 shows the values of information criteria for competitive distributions for data set I. Fig. 6 displays the fitted pdf, cdf, and HRF for data set I for the NATFD.

Data set 2: This data set is generated data to simulate the strengths of glass fibers. The data set is obtained by Mahmoud and Mandouh [29].

1.014, 1.248, 1.267, 1.271, 1.272, 1.275, 1.276, 1.355, 1.361, 1.364, 1.379, 1.409, 1.426, 1.459, 2.456, 2.592, 1.292, 1.081, 1.082, 1.185, 1.223, 1.304, 1.306, 1.46, 1.476, 1.481, 1.484, 1.501, 1.506, 1.524, 1.526, 1.535, 1.541,1.568, 1.735, 1.278, 1.286, 1.288, 1.867, 1.876, 1.878, 1.91, 1.916, 1.972, 2.012, 1.747, 1.748, 1.757, 1.800, 1.579, 1.581, 1.591, 1.593, 1.602, 1.666, 1.67, 1.684, 1.691, 1.704, 1.731, 1.806, 3.197, 4.121. Table 7 shows values of the ML estimates and (standard error in parenthesis) for data set II. Table 8 shows the information criteria values for competitive distributions for data set II.

Table 7
The values of information criteria for competitive distributions for data set I.

Model	−l	AIC	CAIC	BIC	HQIC	K.S statistic	W*	A*	p-value (K.S)
NATFD	53.085	110.17	110.29	115.38	112.28	0.08341	0.09632	0.67625	0.4897
ATFD	53.40	110.80	110.93	116.01	112.91	0.14194	0.48081	0.73348	0.03556
WD	90.149	184.29	184.42	189.50	186.40	0.1949	0.90480	5.6295	0.00099
FD	53.691	111.38	111.50	116.59	113.49	0.08745	0.10905	0.76585	0.4288
EED	57.11	118.23	118.35	123.44	120.34	0.10625	0.12958	1.0305	0.3206
IGD	60.93	125.86	125.98	131.07	127.97	0.12605	0.27139	1.7397	0.08337
BD	74.86	153.73	153.85	158.94	155.83	0.23682	0.22896	1.54825	2.69e−05
GD	68.39	140.79	140.92	146.08	142.90	0.11716	0.38261	2.66515	0.1284
ED	150.55	303.10	303.14	305.70	304.15	0.44387	0.38107	2.6555	2.2e−16
RD	97.91	197.83	197.87	200.43	198.88	0.29294	0.66265	4.29857	7.038e−08
GBD	183.61	369.22	369.27	371.83	370.28	0.62187	0.57716	3.8298	2.2e−16
IRD	89.86	181.72	181.76	184.33	182.78	0.32515	0.08619	0.70538	1.313e−09

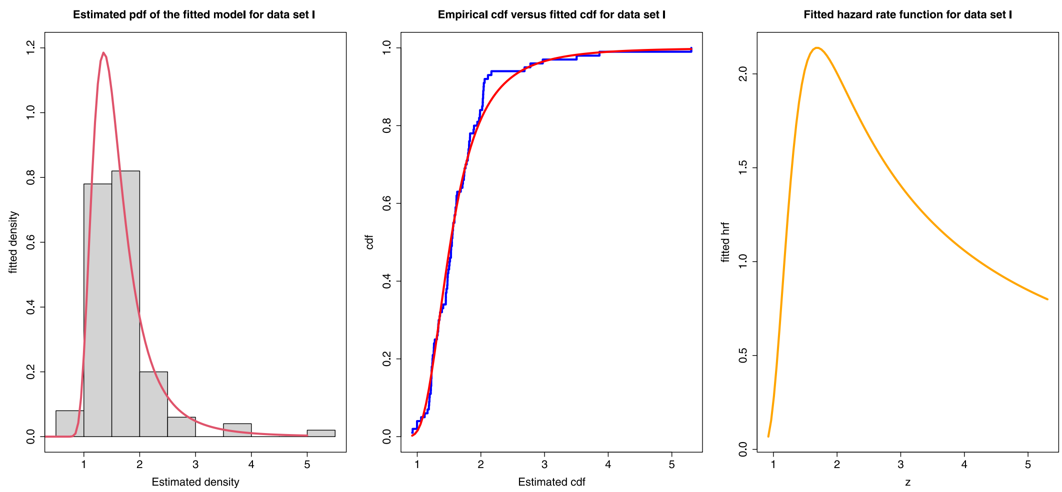

Fig. 6. Fitted pdf, cdf and HRF for data set I NATFD.

Table 8
Values of the ML estimates and (standard error in parenthesis) for data set II.

Model	$\hat{\alpha}$	$\hat{\beta}$
NATFD	6.12036 (1.14570)	5.27478 (0.50916)
ATFD	6.6025 (1.14108)	4.8981 (0.48234)
WD	0.16887 (0.03795)	3.06201 (0.24032)
FD	6.49809 (1.20980)	5.4378 (0.51925)
EED	3.4626 (0.33022)	144.503 (65.9892)
IGD	0.00474 (0.00155)	7.4435 (0.42744)
BD	40.9238 (31.0703)	0.05428 (0.04153)
GD	9.81788 (1.7588)	15.862 (2.7970)
ED	0.61895 (0.07798)	....
RD	0.35226 (0.04438)	....
GBD	0.83166 (0.08715)	....
IRD	2.2326 (0.28128)	....

Fig. 7 displays the fitted pdf, cdf, and HRF for data set II for the NATFD (see Table 9).

10. Conclusions

The novel Arctan-G class of distributions was presented in this study as a new generator, and its applicability was then investigated using the Frechet distribution as the baseline model. We investigate some of its structural features. The model parameters are estimated using the maximum likelihood method. A Monte Carlo simulation analysis is provided to verify the estimates. Then, two real-world data sets and two potent rival models are examined. According to several goodness-of-fit statistics, the suggested model consistently performs better than the others. We anticipate that areas like dependability engineering, healthcare, and engineering data sets will benefit more from the suggested family and the models it produces.

CRediT authorship contribution statement

Getachew Tekle Mekiso: Conceptualization, Data curation, Formal analysis.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Table 9
The values of information criteria for competitive distributions for data set II.

Model	−l	AIC	CAIC	BIC	HQIC	K.S statistic	W*	A*	p-value (k.S)
NATFD	19.712	43.42	43.62	47.71	45.11	0.07280	0.0319	0.47778	0.868
ATFD	19.79	43.59	43.79	47.88	45.28	0.1318	0.09947	0.49968	0.2048
WD	46.366	96.73	96.93	101.02	98.41	0.2051	0.70779	4.3253	0.00841
FD	20.063	44.12	44.32	48.41	45.81	0.07722	0.070681	0.53322	0.8187
EED	22.608	49.21	49.41	53.50	50.90	0.07783	0.10006	0.80449	0.8114
IGD	22.82	49.65	49.85	53.94	51.34	0.10736	0.13478	0.90701	0.432
BD	41.171	86.34	86.54	90.62	88.02	0.31366	0.11918	0.87552	5.246e−06
GD	68.099	66.41	66.61	70.69	68.09	0.13015	0.30560	2.10908	0.2166
ED	93.222	188.44	188.51	190.58	189.28	0.47214	0.30458	2.1029	2.081e−13
RD	56.847	115.69	115.76	117.83	116.53	0.34608	0.47034	3.05028	3.031e−07
GBD	113.66	229.33	229.40	231.48	230.18	0.65032	0.42616	2.81161	6.661e−16
IRD	53.38	108.76	108.82	110.90	109.60	0.36043	0.08650	0.70992	7.762e−08

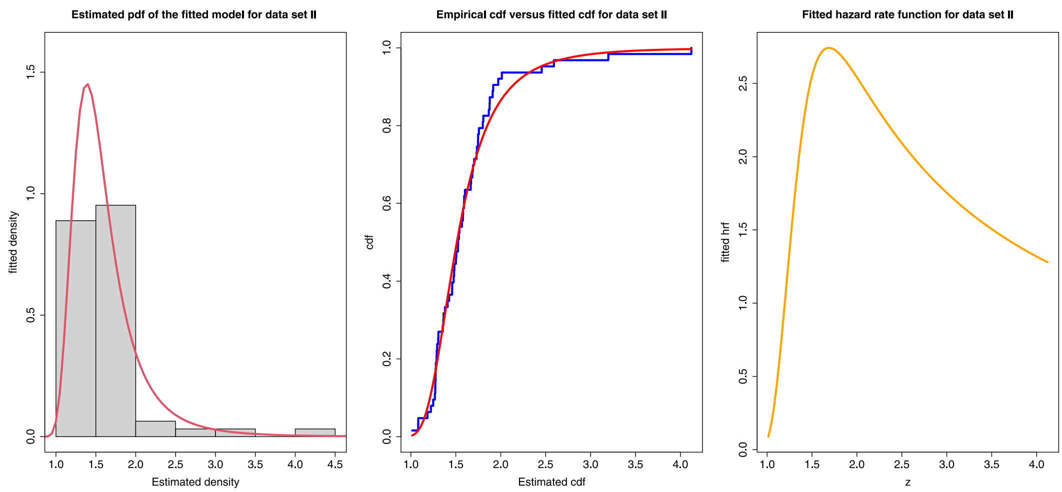

Fig. 7. Fitted pdf, cdf, and HRF for data set II NATFD.

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