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THE WEIGHTED INVERSE AILAMUJIA DISTRIBUTION WITH APPLICATIONS TO REAL LIFE DATA

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ABSTRACT

This work addresses the weighted inverse Ailamujia distribution, which is an expansion of the inverse Ailamujia distribution (WIAD). Several distributional characteristics of the established distribution have been deduced and evaluated. The explicit expression of distinct ageing factors has been achieved. Several plots depict the behaviour of the probability density function (PDF) and the cumulative distribution function (CDF). The parameters of the formed distribution were estimated using the well-known maximum likelihood methodology. Furthermore, the likelihood ratio test of the stated distribution was achieved. Eventually, two examples are offered to exemplify the effectiveness of the formulated model.

KEYWORDS

Inverse Ailamujia distribution, Moments, Order statistics, Reliability measures, weighted distribution, likelihood ratio test.

Mathematics subject classification: 60-XX, 62-XX, 11-KXX

1. INTRODUCTION

In the theory of probability, Ailamujia distribution introduced by Lv et al. (2002), is a newly formulated distribution having several usages in different fields of science including engineering, bio-science and actuarial science. They applied it to analyse real-world data and investigated its numerous statistical features. The Ailamujia distribution's probability density function and cumulative distribution function are described by

$$f\left(y,\theta\right)=4\theta^{2}ye^{-2\theta y}\ ;\ y>0,\theta>0$$

$$F\left(y,\theta\right)=1-\left(1+2\theta y\right)e^{-2\theta y}\ ;\ y>0,\theta>0$$

where θ denotes a scale parameter.

In recent past decade authors have proposed several extensions of Ailamujia distribution. Pan et al. (200), Long (2015), Yu et al. (2015). Recently, Ahmad et al.

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(2020) created an inverse counterpart of the Ailamujia distribution and assessed its applicability in two real scenarios.

Assume Y indicates random variable follows inverse Ailamujia distribution. Then its probability density function (PDF), is given by

$$f(y,\theta)=4\theta^2\frac{1}{y^3}e^{-\frac{2\theta}{y}} \quad ; y>0,\theta>0 \tag{1.1}$$

The related cumulative distribution function (CDF), is given by

$$F(y,\theta)=\frac{(2\theta+y)}{y}e^{-\frac{2\theta}{y}} \quad ; y>0,\theta>0 \tag{1.2}$$

The a^{th} moment of the inverse Ailamujia distribution is given as the following

$$E\left(y^a\right)=\int_0^\infty y^a 4\theta^2\frac{1}{y^3}e^{-\frac{2\theta}{y}}dy$$

$$=4\theta^2\int_0^\infty y^{a-3}e^{-\frac{2\theta}{y}}dy$$

Making the substitution $\frac{2\theta}{y}=z$ so that $-\frac{1}{y^2}dy=\frac{1}{2\theta}dz$, we have

$$E\left(y^a\right)=(2\theta)^a\int_0^\infty z^{1-a}e^{-z}dz$$

$$=(2\theta)^a\,\Gamma(2-a)\,. \tag{1.3}$$

2. WEIGHTED INVERSE AILAMUJIA DISTRIBUTION

In recent past decades several methods have been developed by statisticians for generalizing the classical distributions. The extension of the classical distributions with addition of extra parameters, provide us more flexible distributions which can be applied in different fields of science. The distributions obtained by assigning weight function, is a familiar method by which generalization of the classical distribution can be done. The weighted distributions are extensively used to analyse different types of real life time data, obtained from several areas of science. The weighted distributions have vast applications related to ecology, bio-science, actuarial science and in engineering. The notion of weighted distributions has been traced first from Fisher (1934), to model the ascertainment bias. Rao (1965) later recapitulated Fisher's approach and articulated it in a more broad way to describe statistical data where the classical distributions are insufficient to capture these observations with equal probabilities. Cox and Zelen (1969) pioneered the notion of length-biased sampling (1974). Many investigations into weighted and length-biased distributions have been conducted by researchers. Das et al. (2011), Kersey et al. (2012), Afaq et al. (2014), Abd El-Monsef et al. (2015), Fatima et al. (2015). Mudasir et al. (2017), Hesham et al. (2017), Aijaz et al. (2018), Chesneau et al. (2020), Ahmad et al. (2020) recently addressed a weighted counterpart of the inverse Maxwell distribution and investigated its various statistical features. We suggest a novel distribution in this study that is an extension of the inverse Ailamujia distribution, known colloquially as the weighted inverse Ailamujia distribution (WIAD).

Definition:

Suppose Y denotes random variable with PDF $f(y)$, then PDF of weighted variable Y_w is define by

$$f_w(y) = \frac{w(y)\, f(y)}{E\left[w(y)\right]};\, y > 0 \tag{2.1}$$

where $w(y)$ is a non–negative weight function and $E\left[w(y)\right] = \int_0^\infty w(y) f(y)\, dy < \infty$.

Let us suppose the weight function $w(y) = y^a$ for this distribution.

Theorem 2.1:

Suppose y denotes random variable follows IAD with PDF $f(y)$. Then the p.d.f of weighted inverse Ailamujia distribution $Y \sim WIAD(\theta, a)$ is given by

$$f_w(y, \theta, a) = \frac{(2\theta)^{2-a}}{\Gamma(2-a)} y^{a-3} e^{\frac{-2\theta}{y}} \quad ; y > 0, \theta > 0, a \neq 2 \tag{2.2}$$

where θ and a are scale and weight parameter respectively.

Proof:

The probability density function of weighted inverse Ailamujia distribution can be obtained by using the equations (1.1), (1.2) and (1.3) in equation (2.1), we get

$$\begin{aligned} f_w(y, \theta, a) &= \frac{y^a}{(2\theta)^a\, \Gamma(2-a)} 4\theta^2 \frac{1}{y^3} e^{\frac{-2\theta}{y}} \\ &= \frac{(2\theta)^{2-a}}{\Gamma(2-a)} y^{a-3} e^{\frac{-2\theta}{y}} \quad ; y > 0, \theta > 0, a \neq 2 \end{aligned}$$

Figure (2.1) and (2.2), illustrates the behaviour of PDF of WIAD for varying values of the parameters.

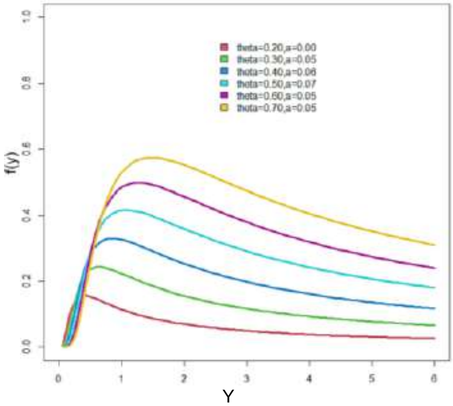

Figure 2.1: PDF of WIAD under Different Values of Parameters

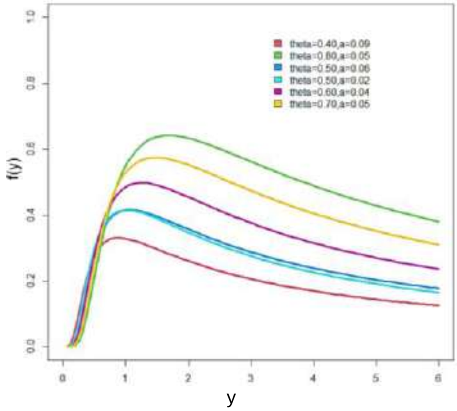

Figure 2.2: PDF of WIAD under Different Values of Parameters

Special Cases of Weighted Inverse Ailamujia Distribution:

1. When $a=0$, The weighted inverse Ailamujia distribution is equivalent to the inverse Ailamujia distribution with probability density function as

$$f\left(y,\theta\right)=4\theta^2\frac{1}{y^3}e^{-\frac{2\theta}{y}}.$$

2. When $a=1$, The weighted inverse Ailamujia distribution is equivalent to the length-biased inverse Ailamujia distribution with probability density function as

$$f\left(y,\theta\right)=2\theta\frac{1}{y^2}e^{-\frac{2\theta}{y}}.$$

Theorem 2.2:

Suppose y denotes random variable follows WIAD distribution. Then the cumulative distribution function of the WIAD distribution with parameters θ and a can be defined as

$$F_w\left(y,\theta,a\right)=\frac{\Gamma\left(2-a,\frac{2\theta}{y}\right)}{\Gamma\left(2-a\right)} \tag{2.3}$$

where $\Gamma\left(s,y\right)=\int_y^\infty t^{s-1}e^{-t}dt$ is an upper incomplete gamma function.

Proof:

The cumulative distribution function of a distribution is defined as

$$F_w\left(y\right)=\int_0^y f\left(y\right)dy \tag{2.4}$$

Substituting equation (2.2) in equation (2.4), we have

$$F_w(y) = \int_0^y \frac{(2\theta)^{2-a}}{\Gamma(2-a)} y^{a-3} e^{\frac{-2\theta}{y}} dy$$

Making substitution $\frac{2\theta}{y} = z$ so that, $-\frac{2\theta}{y^2} dy = dz$, $\frac{2\theta}{y} < z < \infty$

$$F_w(y) = \frac{1}{\Gamma(2-a)} \int_{\frac{2\theta}{y}}^{\infty} z^{(2-a)-1} e^{-z} dz$$

$$F_w(y) = \frac{\Gamma\left(2-a, \frac{2\theta}{y}\right)}{\Gamma(2-a)}$$

Figure (2.3) and (2.4), illustrates the behaviour of c.d.f of WIAD for varying values of the parameters.

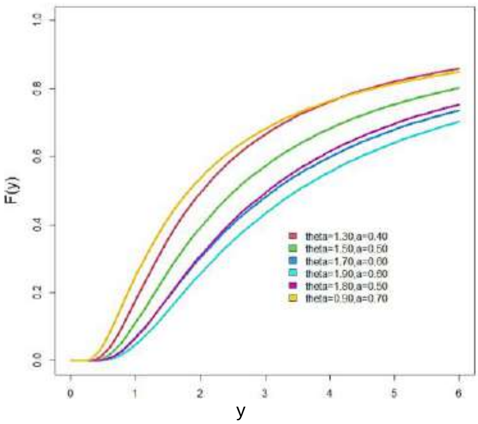

Figure 2.3: CDF of WIAD under Different Values of Parameters

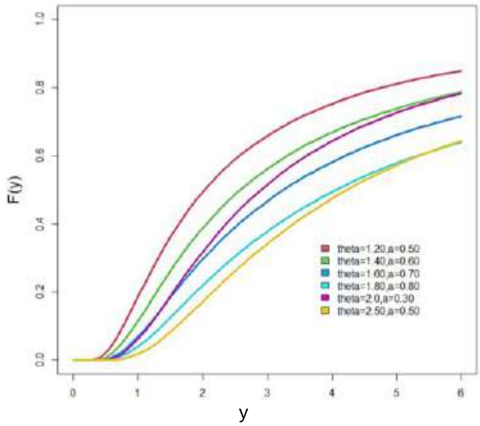

Figure 2.3: CDF of WIAD under Different Values of Parameters

3. RELIABILITY MEASURES

Theorem 3.1:

Suppose y denotes random variable follows WIAD distribution with parameter θ and a . The survival function of the WIAD distribution can be written as.

$$S_w(y) = \frac{\gamma\left(2-a, \frac{2\theta}{y}\right)}{\Gamma(2-a)}$$

where $\gamma(s, y) = \int_0^y t^{s-1} e^{-t} dt$ is lower incomplete gamma function.

Proof:

The survival function of the random variable y is defined as

$$S_w(y) = 1 - F_w(y) \tag{3.1}$$

Using equation (2.3) in (3.1), survival function of WIAD distribution can be obtained as

$$S_w(y) = 1 - \frac{\Gamma\left(2-a, \frac{2\theta}{y}\right)}{\Gamma(2-a)}$$

$$= \frac{\Gamma(2-a) - \Gamma\left(2-a, \frac{2\theta}{y}\right)}{\Gamma(2-a)}$$

$$= \frac{\gamma\left(2-a, \frac{2\theta}{y}\right)}{\Gamma(2-a)} \tag{3.2}$$

Figure (3.1) and (3.2), illustrates the behaviour of survival function of WIAD for varying values of the parameters.

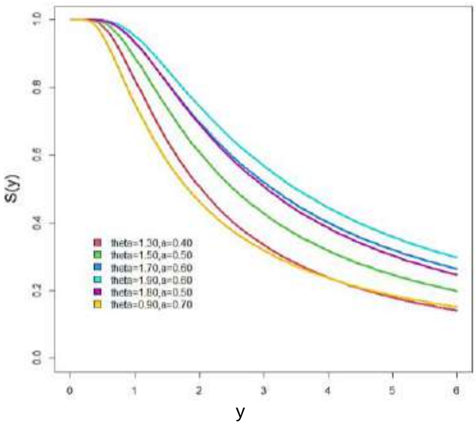

Figure 3.1: Survival Function of WIAD under Different Values of Parameters

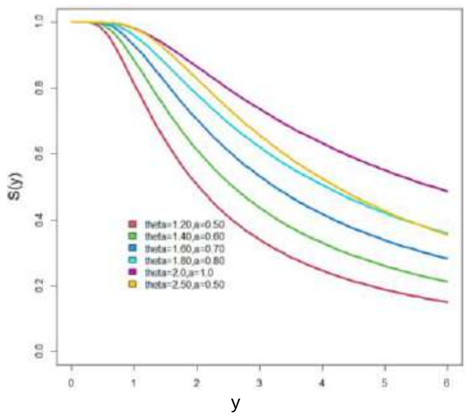

Figure 3.2: Survival Function of WIAD under Different Values of Parameters

Theorem 3.2:

Suppose y denotes a random variable follows WIAD distribution with parameter θ and a. The hazard rate function of the WIAD distribution can be expressed as.

$$H_w(y) = \frac{(2\theta)^{2-a}\, y^{a-3} e^{-\frac{2\theta}{y}}}{\gamma\left(2-a, \frac{2\theta}{y}\right)}$$

Proof:

Suppose y denotes random variable follows WIAD distribution with PDF $f(y)$ and survival function $S(y)$, then the hazard rate function is defined as

$$H_w(y) = \frac{f_w(y)}{S_w(y)} \tag{3.3}$$

Substituting value of equations (2.2), (3.2), into equation (3.3), we have

$$H_w(y) = \frac{(2-a)^{2-a}\, y^{a-3} e^{-\frac{2\theta}{y}} \Big/ \Gamma(2-a)}{\gamma\left(2-a, \frac{2\theta}{y}\right) \Big/ \Gamma(2-a)}$$

$$= \frac{(2-a)^{2-a}\, y^{a-3} e^{-\frac{2\theta}{y}}}{\gamma\left(2-a, \frac{2\theta}{y}\right)}$$

Figure (3.3) and (3.4), illustrates the behaviour of hazard rate function of WIAD for varying values of the parameters.

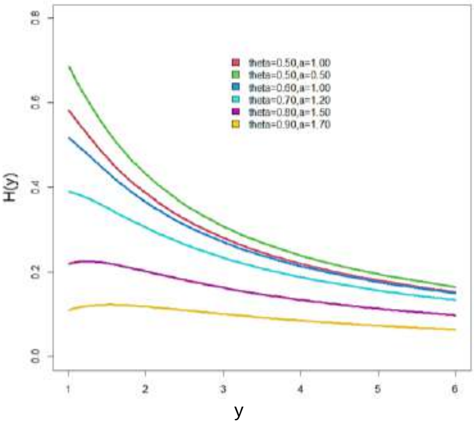

Figure 3.3: HRF of WIAD under Different Values of Parameters

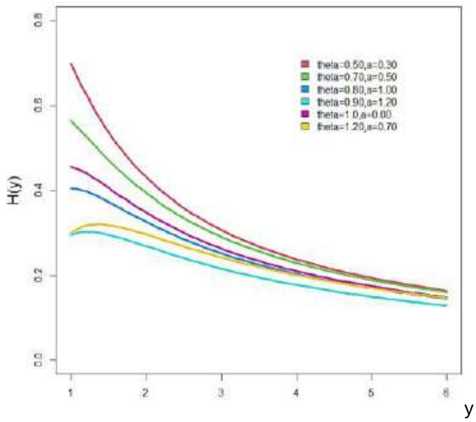

Figure 3.4: HRF of WIAD under Different Values of Parameters

Theorem 3.3:

Suppose y denotes random variable follows WIAD distribution with parameter θ and a. The reverse hazard rate function of the WIAD distribution can be expressed as.

$$h_r(y) = \frac{(2\theta)^{2-a}\, y^{a-3} e^{-\frac{2\theta}{y}}}{\Gamma\left(2-a, \frac{2\theta}{y}\right)}$$

Proof:

Suppose y denotes random variable follows WIAD distribution with p.d.f $f_w(y)$ and survival function $S(y)$, then the reverse hazard rate function is defined as

$$h_r(y) = \frac{f_w(y)}{F_w(y)} \tag{3.4}$$

Substituting equation (2.2) and (2.3), into (3.4), we have

$$h_r(y) = \frac{(2\theta)^{2-a} y^{a-3} e^{-\frac{2\theta}{y}} \Big/ \Gamma(2-a)}{\Gamma\left(2-a, \frac{2\theta}{y}\right) \Big/ \Gamma(2-a)}$$

$$= \frac{(2\theta)^{2-a} y^{a-3} e^{-\frac{2\theta}{y}}}{\Gamma\left(2-a, \frac{2\theta}{y}\right)}$$

Figure (3.5) and (3.6), illustrates the behaviour of reverse hazard rate function of WIAD for varying values of the parameters.

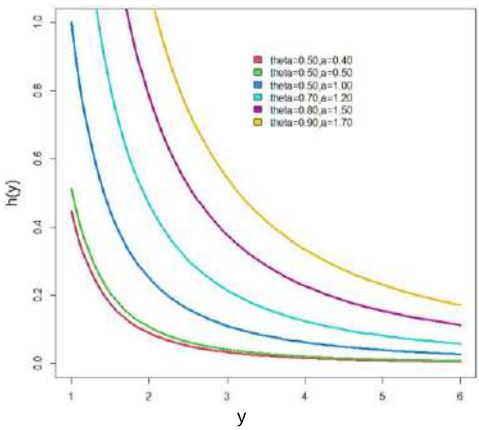

Figure 3.5: RHRF of WIAD under Different Values of Parameters

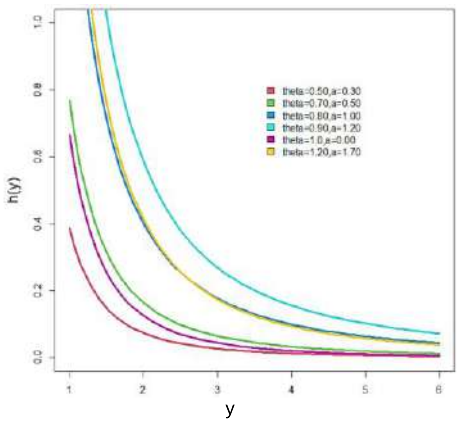

Figure 3.6: RHRF of WIAD under Different Values of Parameters

4. STATISTICAL PROPERTIES OF THE WIAD DISTRIBUTION

This part addresses the weighted inverse Ailamujia distribution's substantial features, including moments, the moment generating function, mode, median, and the harmonic mean.

4.1 Moments of the WIAD distribution

Theorem 4.1:

If $Y \sim WIAD(a,\theta)$, then r^{th} moment of a continuous random variable y is given as

$$\mu_r = E(Y^r) = \frac{(2\theta)^r \Gamma(2-(r+a))}{\Gamma(2-a)}$$

Proof:

Let Y denotes random variable follows WIAD distribution. Then r^{th} moment denoted by μ_r is given as

$$\mu_r = E(Y^r) = \int_0^\infty y^r f_w(y,\theta,a)dy$$

$$= \int_0^\infty y^r \frac{(2\theta)^{2-a}}{\Gamma(2-a)} y^{a-3} e^{-\frac{2\theta}{y}} dy$$

$$= \frac{(2\theta)^{2-a}}{\Gamma(2-a)} \int_0^\infty y^{r+a-3} e^{-\frac{2\theta}{y}} dy$$

Making substitution $\frac{2\theta}{y} = z$ so that, $-\frac{2\theta}{y^2} dy = dz$

After solving the integral, we get

$$\mu_r = E(Y^r) = \frac{(2\theta)^r \Gamma(2-(r+a))}{\Gamma(2-a)} \tag{4.1}$$

Now substituting $r = 1$

$$\mu = E(Y) = \frac{2\theta}{1-a}$$

For $r = 2$ moment does not exists.

Theorem 4.2:

If $Y \sim WIAD(a,\theta)$, then moment generating function of a continuous random variable y is given as.

$$M_Y(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \frac{(2\theta)^r \Gamma(2-(r+a))}{\Gamma(2-a)}$$

Proof:

Let Y denotes random variable follows WIAD distribution. Then moment generating function denoted as $M_Y(t)$ is given by

$$M_Y(t) = E\left(e^{ty}\right) = \int_0^\infty e^{ty} f_w(y,\theta,a)\,dy$$

Using Taylor's series

$$M_Y(t) = \int_0^\infty \left(1 + ty + \frac{(ty)^2}{2!} + \ldots\right) f_w(y,\theta,a)\,dy$$

$$= \sum_{r=0}^\infty \frac{t^r}{r!} \int_0^\infty y^r f_w(y,\theta,a)\,dy$$

$$= \sum_{r=0}^\infty \frac{t^r}{r!} \frac{(2\theta)^r \Gamma(2-(r+a))}{\Gamma(2-a)}$$

Theorem 4.3:

If $Y \sim WIAD(a,\theta)$, then characteristics function of a continuous random variable y is given as

$$\varphi_Y(t) = \sum_{r=0}^\infty \frac{(it)^r}{r!} \frac{(2\theta)^r \Gamma(2-(r+a))}{\Gamma(2-a)}$$

Proof:

Let Y denotes random variable follows WIAD distribution. Then characteristics function denoted as $\varphi_Y(t)$ is given by

$$\varphi_Y(t) = E\left(e^{ity}\right) = \int_0^\infty e^{ity} f_w(y,\theta,a)\,dy$$

Using Taylor's series

$$\varphi_Y(t) = \int_0^\infty \left(1 + ity + \frac{(ity)^2}{2!} + \ldots\right) f_w(y,\theta,a)\,dy$$

$$= \sum_{r=0}^\infty \frac{(it)^r}{r!} \int_0^\infty y^r f_w(y,\theta,a)\,dy$$

$$= \sum_{r=0}^\infty \frac{(it)^r}{r!} \frac{(2\theta)^r \Gamma(2-(r+a))}{\Gamma(2-a)}$$

4.4 Mode and Median of Inverse Ailamujia Distribution

Taking logarithm of equation (2.2), we have

$$\log f_w(y,\theta,a) = (2-a)\log(2\theta) - \log\Gamma(2-a) + (a-3)\log y - \frac{2\theta}{y} \tag{4.2}$$

Differentiate equation (4.2), with respect to y, we obtain

$$\frac{\partial \log f_w(y,\theta,a)}{\partial y} = \frac{a-3}{y} + \frac{2\theta}{y^2}$$

Substituting $\frac{\partial \log f_w(y,\theta,a)}{\partial y} = 0$, we get

$$y = \frac{2\theta}{a-3} \Rightarrow M_0 = y_0 = \frac{2\theta}{a-3}$$

Using the empirical formula for median, we get

$$M_d = \frac{1}{3}M_0 + \frac{2}{3}\mu_1'$$

$$M_d = \frac{1}{3}.\frac{2\theta}{(a-3)} + \frac{2}{3}.\frac{2\theta}{(1-a)} = \frac{2\theta(a-5)}{(a-3)(1-a)}$$

4.5 Harmonic Mean of Weighted Inverse Ailamujia Distribution

$$\frac{1}{H} = E\left(\frac{1}{Y}\right) = \int_0^\infty \frac{1}{y} f_w(y,\theta,a)\,dy$$

$$= \frac{(2\theta)^{2-a}}{\Gamma(2-a)} \int_0^\infty y^{a-3} e^{-\frac{2\theta}{y}}\,dy$$

Making substitution $\frac{2\theta}{y} = z$ so that $\frac{1}{y^2}dy = -\frac{1}{2\theta}dz$

$$\frac{1}{H} = \frac{(2\theta)^{2-a}}{\Gamma(2-a)} \int_0^\infty \left(\frac{2\theta}{z}\right)^{a-2} e^{-z} \frac{1}{2\theta}\,dz$$

$$= \frac{1}{2\theta\Gamma(2-a)} \int_0^\infty z^{2-a} e^{-z}\,dz$$

$$\frac{1}{H} = \frac{\Gamma(3-a)}{2\theta\Gamma(2-a)} = \frac{(2-a)}{2\theta}$$

5. SHANNON'S ENTROPY OF WEIGHTED INVERSE AILAMUJIA DISTRIBUTION

Shannon established the idea of entropy in 1948. The entropy is stated as the average rate at which information is created by a random source of data and is described as

$$H(y) = -E\left[\log f(y)\right] = -\int_0^\infty \left[\log f(y)\right] f(y)\, dy$$

Provided the integral is convergent.

Theorem 5.1:

Suppose $\underset{-}{y} = (y_1, y_2, ... y_n)$ be n positive identical independently distributed (iid) random samples selected from a population with a weighted inverse Ailamujia distribution. Shannon's entropy is therefore expressed as

$$H(y) = \log\left[\frac{\Gamma(2-a)}{(2\theta)^{2-a}}\right] - 2\theta(a-3)\left[\log 2\theta - \psi(2-a)\right] + (2-a)$$

where $\psi(.)$ represents digamma function

Proof:

Shannon's entropy is defined as

$$H_w(y,\theta,a) = -E\left[\log f_w(y,\theta,a)\right]$$

$$= -E\left[\log\left(\frac{2\theta}{\Gamma(2-a)} y^{a-3} e^{-\frac{2\theta}{y}}\right)\right]$$

$$= -\log\left(\frac{(2\theta)^{2-a}}{\Gamma(2-a)}\right) - (a-3) E\left[\log y\right] + 2\theta E\left(\frac{1}{y}\right) \quad (5.1)$$

Now

$$E(\log y) = \int_0^\infty (\log y) f_w(y,\theta,a)\, dy$$

$$= \frac{(2\theta)^{2-a}}{\Gamma(2-a)} \int_0^\infty (\log y) y^{a-3} e^{-\frac{2\theta}{y}} dy$$

Making substitution $\frac{2\theta}{y} = z$ so that $\frac{1}{y^2} = -\frac{1}{2\theta} dz$

$$= \frac{(2\theta)^{2-a}}{\Gamma(2-a)} \int_0^\infty \log\left(\frac{2\theta}{z}\right)\left(\frac{2\theta}{z}\right)^{a-1} e^{-z} dz$$

$$= \frac{2\theta}{\Gamma(2-a)} \int_0^\infty (\log 2\theta - \log z) z^{1-a} e^{-z} dz$$

After solving the integral, we get

$$= 2\theta\left[\log 2\theta - \psi(2-a)\right] \tag{5.2}$$

where $\psi(.)$ represents digamma function

Also

$$E\left(\frac{1}{y}\right) = \int_0^\infty \frac{1}{y} f_w(y,\theta,a)\, dy$$

$$= \frac{(2\theta)^{2-a}}{\Gamma(2-a)} \int_0^\infty y^{a-4} e^{-\frac{2\theta}{y}} dy$$

Making substitution $\frac{2\theta}{y} = z$ so that $\frac{1}{y^2} dy = -\frac{1}{2\theta} dz$

$$= \frac{(2\theta)^{2-a}}{\Gamma(2-a)} \int_0^\infty \left(\frac{2\theta}{z}\right)^{a-z} e^{-z} \frac{1}{2\theta} dz$$

After solving the integral, we get

$$E\left(\frac{1}{y}\right) = \frac{2-a}{2\theta} \tag{5.3}$$

Substituting value of equations (5.3), (5.2) in equation (5.1), we get

$$H(y) = \log\left[\frac{\Gamma(2-a)}{(2\theta)^{2-a}}\right] - 2\theta(a-3)\left[\log 2\theta - \psi(2-a)\right] + (2-a).$$

6. ORDER STATISTICS OF WEIGHTED INVERSE AILAMUJIA DISTRIBUTION

If $Y_1, Y_2, \ldots Y_n$ denotes random samples from weighted inverse of Ailamujia distribution. Let us suppose $Y_{(1)} < Y_{(2)} < \ldots < Y_{(n)}$ denotes its order statistics. Then the p.d.f of the r^{th} order statistics of the weighted inverse Ailamujia distribution, say $X = Y_{(r)}$

$$f_X(x) = \frac{n!}{(r-1)!(n-r)!} F^{r-1}(x)\left[1-F(x)\right]^{n-r} f(x)$$

$$= \frac{n!}{(r-1)!(n-r)!} \sum_{j=0}^{\infty} \binom{n-r}{j} (-1)^j F^{r+j-1}(x) f(x)$$

$$= \sum_{j=0}^{\infty} \binom{n-r}{j} (-1)^j \left[\frac{\Gamma\left(2-a, \frac{2\theta}{y}\right)}{\Gamma(2-a)} \right]^{r+j-1} \frac{(2\theta)^{2-a}}{\Gamma(2-a)} y^{a-3} e^{\frac{-2\theta}{y}}$$

The corresponding c.d.f of X is given by

$$F_X(x) = \sum_{i=r}^{n} F^i(x)\left[1-F(x)\right]^{n-i}$$

$$F_X(x) = \sum_{i=r}^{n} \sum_{j=0}^{n-j} \binom{n}{i} \binom{n-i}{j} (-1)^j F^{i+j}(x)$$

$$= \sum_{i=r}^{n} \sum_{j=0}^{n-i} \binom{n}{i} \binom{n-i}{j} (-1)^j \left[\frac{\Gamma\left(2-a, \frac{2\theta}{y}\right)}{\Gamma(2-a)} \right]^{i+j}.$$

7. ESTIMATION OF PARAMETERS OF WEIGHTED INVERSE AILAMUJIA DISTRIBUTION

This section provides different approaches of parameter estimation of weighted Ailamujia distribution

7.1 Methods of Moments

To determine the sample moments of the weighted inverse Ailamujia distribution, we compare the population and sample moments.

$$\mu_1 = \frac{1}{n} \sum_{i=0}^{n} y_i$$

$$\bar{Y} = \frac{2\theta}{1-a} \Rightarrow \hat{\theta} = \bar{Y} \frac{(1-a)}{2}$$

7.2 Maximum Likelihood Estimation

Let $Y_1, Y_2, ..., Y_n$ denotes random sample of size n from weighted inverse Ailamujia distribution then its likelihood function is given by

$$l = \prod_{i=1}^{n} f_w\left(y, \theta, a\right)$$
$$= \prod_{i=1}^{n} \frac{\left(2\theta\right)^{2-a}}{\Gamma\left(2-a\right)} y_i^{a-3} e^{-\frac{2\theta}{y_i}}$$
$$= \frac{\left(2\theta\right)^{n(2-a)}}{\left(\Gamma\left(2-a\right)\right)^n} \prod_{i=1}^{n} y_i^{a-3} e^{-2\theta \sum_{i=1}^{n} \frac{1}{y_i}}$$

The log likelihood function is

$$\log l = n\left(2-a\right)\log\left(2\theta\right) - n\log\Gamma\left(2-a\right) + \left(a-3\right)\sum_{i=1}^{n}\log y_i - 2\theta\sum_{i=1}^{n}\frac{1}{y_i} \qquad (7.1)$$

Differentiate w.r.t parameters θ and a, we have

$$\frac{\partial \log l}{\partial \theta} = \frac{n\left(2-a\right)}{2\theta} - 2\sum_{i=1}^{n}\frac{1}{y_i} \qquad (7.2)$$

$$\frac{\partial \log l}{\partial a} = -n\log\left(2\theta\right) - n\psi\left(2-a\right) + \sum_{i=1}^{n} y_i \qquad (7.3)$$

where $\psi\left(2-a\right) = \frac{\Gamma'\left(2-a\right)}{\Gamma\left(2-a\right)}$ is digamma function

The equations (7.2) and (7.3) cannot be expressed in compact form. Therefore $\hat{\theta}$ and $\hat{a}$ cannot be yield in compact form. So we can apply iterative methods such as Newton–Raphson method, secant method, Regula-falsi method etc. The MLE of the parameters denoted as $\hat{\varsigma}\left(\hat{\theta}, \hat{a}\right)$ of $\varsigma\left(\theta, a\right)$ can be obtained by using the above methods.

Since the MLE of $\hat{\varsigma}$ follows asymptotically normal distribution as given as follows

$$\sqrt{n}\left(\hat{\varsigma} - \varsigma\right) \rightarrow N\left(0, I^{-1}\left(\varsigma\right)\right)$$

where $I^{-1}\left(\varsigma\right)$ is the limiting variance covariance matrix $\hat{\varsigma}$ and $I^{-1}\left(\varsigma\right)$ is a 2x2 Fisher Information matrix i.e.

$$I^{-1}\left(\varsigma\right) = -\frac{1}{n}\begin{bmatrix} E\left(\frac{\partial^2 \log l}{\partial \theta^2}\right) & E\left(\frac{\partial^2 \log l}{\partial a \partial \theta}\right) \\ E\left(\frac{\partial^2 \log l}{\partial \theta \partial a}\right) & E\left(\frac{\partial^2 \log l}{\partial a^2}\right) \end{bmatrix}$$

where

$$\frac{\partial^2 \log l}{\partial \theta^2} = \frac{-n(2-a)}{2\theta^2}$$

$$\frac{\partial^2 \log l}{\partial a^2} = -n\psi'(2-a) \ , \ \frac{\partial^2 \log l}{\partial \theta \partial a} = \frac{\partial^2 \log l}{\partial a \partial \theta} = \frac{-n}{\theta}$$

Hence the approximate $100(1-\psi)$ % confidence interval for θ and a are respectively given as

$$\theta \pm Z_{\frac{\psi}{2}} \sqrt{I_{\theta\theta}^{-1}(\hat{\varsigma})} \quad \hat{a} \pm Z_{\frac{\psi}{2}} \sqrt{I_{aa}^{-1}(\hat{\varsigma})}$$

where $Z_{\frac{\psi}{2}}$ denotes the ψ^{th} percentile of the standard distribution.

8. LIKELIHOOD RATIO TEST OF WEIGHTED INVERSE AILAMUJIA DISTRIBUTION

Suppose $Y_1, Y_2, ..., Y_n$ are random samples from weighted inverse Ailamujia distribution. We use the hypothesis

$$H_0 : f(y) = f(y,\theta) \text{ Against } H_1 : f(y) = f_w(y,\theta,a)$$

Now we examine whether the random sample of size n comes from inverse Ailamujia distribution or the weighted inverse Ailamujia distribution. Then the following test statistics is used

$$\Lambda = \frac{L_1}{L_0} = \prod_{i=1}^{n} \frac{f_w(y,\theta,a)}{f(y,\theta)}$$

$$\Lambda = \frac{L_1}{L_0} = \prod_{i=1}^{n} \frac{\frac{(2\theta)^{(2-a)}}{\Gamma(2-a)} y_i^{(a-3)} e^{-\frac{2\theta}{y_i}}}{4\theta^2 y_i^{-3} e^{-\frac{2\theta}{y_i}}}$$

$$= \frac{(2\theta)^{n(2-a)}}{\left[4\theta^2 \Gamma(2-a)\right]^n} \prod_{i=1}^{n} y_i^a$$

We reject null hypothesis if

$$\Lambda = \left[\frac{1}{(2\theta)^a \Gamma(2-a)}\right]^n \prod_{i=1}^{n} y_i^a > c$$

$$\Lambda^* = \prod_{i=1}^{n} y_i^a > c^* \text{, where } c^* = c\left\{\left(2\theta\right)^a \Gamma\left(2-a\right)\right\}^n$$

For a large sample of size n, $2\log\Lambda$ is distributed as chi-square distribution with one degree of freedom. Thus p-value is obtained from chi-square distribution. Also we reject the null hypothesis, when probability value is given by

$$P\left(\Lambda^* > b^*\right),$$

where $b^* = \prod_{i=1}^{n} y_i$ is less than a specified level of significance, where $\prod_{i=1}^{n} y_i$ is the observed value of the statistics Λ^*.

9. DATA ANALYSIS

This section represents applicability of the explored distribution by considering a real life time data set. The formulated distribution is compared with length biased inverse Ailamujia distribution (LIAD), inverse Ailamujia distribution (IAD), inverse power Rayleigh distribution, weighted inverse Rayleigh distribution (WIRD), inverse Rayleigh distribution (IRD), inverse Lindley distribution (ILD) and area biased Ailamujia distribution (ABAD).

Data Set I:

The below actual data set depicts the 72 exceedances of flood heights (in m3/s) of the Wheaton River near Carcross in the Yukon Territory from 1958 to 1984 (rounded to one decimal point). Merovci and Puka (2014) recently investigated this data and incorporated it into a transmuted Pareto distribution.

1.7, 2.2, 14.4, 1.1, 0.4, 20.6, 5.3, 0.7, 1.9, 13.0, 12.0, 9.3, 1.4, 18.7, 8.5, 25.5, 11.6, 14.1, 22.1, 1.1, 2.5, 14.4, 1.7, 37.6, 0.6, 2.2, 39.0, 0.3, 15.0, 11.0, 7.3, 22.9, 1.7, 0.1, 1.1, 0.6, 9.0, 1.7, 7.0, 20.1, 0,4, 2.8, 14.1, 9.9, 10.4, 10.7, 30.0, 3.6, 5.6, 30.8, 13.3, 4.2, 25.5, 3.4, 11.9, 21.5, 27.6, 36.4, 2.7, 64.0, 1.5, 2.5, 27.4, 1.0, 27.1, 20.2, 16.8, 5.3, 9.7, 27.5, 2.5, 7.0

The description of the data is presented in Table 9.1 and 9.3. The estimates of the parameters, log-likelihood, and Akaike information criteria (AIC) etc. for the data set are generated and presented in Table 9.2 and 9.4.

Table 9.1
Data Description for the First Data Set I

Minimum	1st Qu.	Median	Mean	3rd Qu.	Maximum
0.100	2.125	9.150	11.926	19.050	64.00

Table 9.2
Performance of Distributions for Real Data Set I

Model	Parameter Estimation	Standard Error	$-2\ln l$	AIC	CAIC	BIC
WIAD	$\hat{\theta}=0.5186$ $\hat{a}=1.4515$	$\hat{\theta}=0.1095$ $\hat{a}=0.07610$	542.33	546.33	546.51	550.88
LBIAD	$\hat{\theta}=0.9457$	$\hat{\theta}=0.1114$	564.66	566.66	566.71	568.93
IAD	$\hat{\theta}=1.8915$	$\hat{\theta}=0.1576$	673.47	675.47	675.52	677.74
IPRD	$\hat{\theta}=0.5821$ $\hat{a}=0.5620$	$\hat{\theta}=0.0840$ $\hat{a}=0.0468$	849.10	853.10	853.28	857.66
WIRD	$\hat{\theta}=0.2150$ $\hat{a}=1.5847$	$\hat{\theta}=0.0620$ $\hat{a}=0.0531$	652.23	656.23	656.40	660.78
IRD	$\hat{\theta}=0.5179$	$\hat{\theta}=0.0863$	714.23	716.23	716.29	718.51
ILD	$\hat{\theta}=2.4412$	$\hat{\theta}=0.2351$	576.31	578.31	578.36	580.58
ABAD	$\hat{\theta}=0.1676$	$\hat{\theta}=0.0098$	694.60	696.60	696.65	698.87

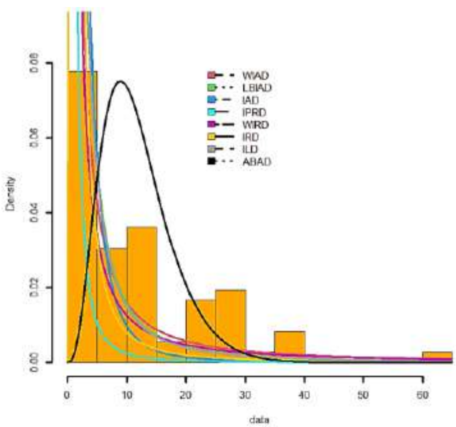

Figure 9.1: Estimated pdf's of the Fitted Models for Data Set I

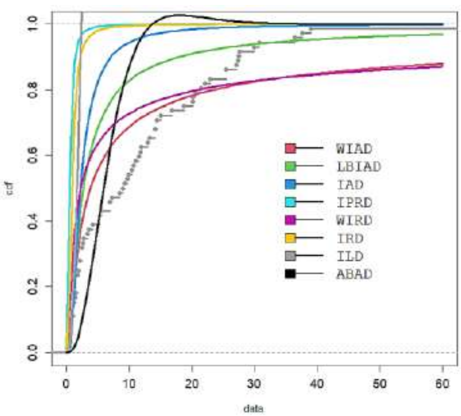

Figure 9.2: Estimated cdf's Versus Fitted cdf's for Data Set I

Data Set II:

The data indicates the remission periods (in months) of a random sample of 128 bladder cancer patients based on Rady et al.'s (2016). Lee and Wang (2003) had already used it. The information is summarised as follows.

0.08,0.20, 0.40, 0.50, 0.51, 0.81, 0.90, 1.05, 1.19, 1.26, 1.35, 1.40, 1.46,1.76, 2.02, 2.02, 2.07, 2.09, 2.23, 2.26, 2.46, 2.54, 2.62, 2.64, 2.69, 2.69, 2.75, 2.83, 2.87, 3.02, 3.25, 3.31, 3.36, 3.36, 3.48, 3.52, 3.57, 3.64, 3.70,3.82, 3.88, 4.18, 4.23, 4.26, 4.33, 4.34, 4.40, 4.50, 4.51, 4.87, 4.98, 5.06, 5.09, 5.17, 5.32, 5.32, 5.34, 5.41, 5.41, 5.49, 5.62, 5.71, 5.85, 6.25, 6.54,6.76, 6.93, 6.94, 6.97, 7.09, 7.26, 7.28, 7.32, 7.39, 7.59, 7.62, 7.63, 7.66, 7.87, 7.93, 8.26, 8.37, 8.53, 8.65, 8.66, 9.02, 9.22, 9.47, 9.74, 10.06, 10.34,10.66, 10.75, 11.25, 11.64, 11.79, 11.98, 12.02, 12.03, 12.07, 12.63, 13.11, 13.29, 13.80, 14.24, 14.76, 14.77, 14.83, 15.96, 16.62, 17.12, 17.14, 17.36, 18.10, 19.13, 20.28, 21.73,22.69, 23.63, 25.74, 25.82, 26.31, 32.15, 34.26, 36.66, 43.01, 46.12, 79.05.

Table 9.3
Data Description for the first Data Set II

Minimum	1^{st} Qu.	Median	Mean	3^{rd} Qu.	Maximum
0.080	3.348	6.395	9.366	11.838	79.050

Table 9.4
Performance of Distributions for Real Data Set II

Model	Parameter Estimation	Standard Error	$-2\ln l$	AIC	CAIC	BIC
WIAD	$\hat{\theta}=0.8877$ $\hat{a}=1.2854$	$\hat{\theta}=0.1326$ $\hat{a}=0.0762$	909.86	913.86	913.95	919.56
LBIAD	$\hat{\theta}=1.2423$	$\hat{\theta}=0.1098$	920.76	922.76	922.79	925.61
IAD	$\hat{\theta}=2.4847$	$\hat{\theta}=0.1552$	1037.7	1039.7	1039.7	1042.6
IPRD	$\hat{\theta}=0.5821$ $\hat{a}=0.5620$	$\hat{\theta}=0.0840$ $\hat{a}=0.0468$	1450.4	1454.4	1454.5	1460.1
WIRD	$\hat{\theta}=0.2715$ $\hat{a}=1.5600$	$\hat{\theta}=0.0574$ $\hat{a}=0.0424$	1140.4	1144.4	1144.5	1150.1
IRD	$\hat{\theta}=0.6173$	$\hat{\theta}=0.0771$	1234.0	1236.0	1236.0	1238.8
ILD	$\hat{\theta}=3.0919$	$\hat{\theta}=0.2286$	936.90	938.90	938.94	941.76
ABAD	$\hat{\theta}=0.2135$	$\hat{\theta}=0.0094$	1007.2	1009.21	1009.24	1012.0

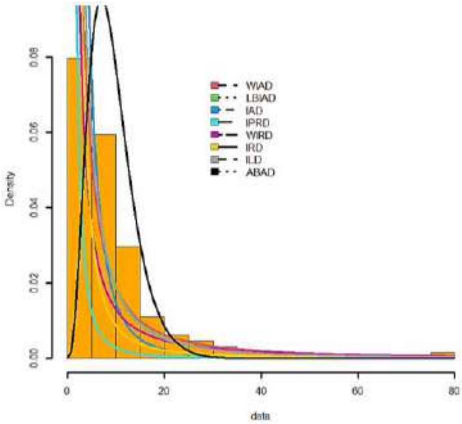

Figure 9.3: Estimated pdf's of the Fitted Models for Data Set II

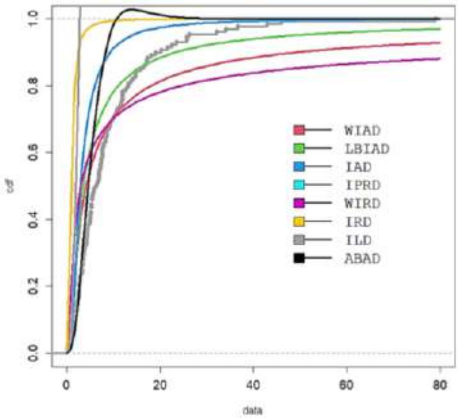

Figure 9.4: Estimated cdf Versus Fitted cdf's for Data Set II

11. CONCLUDING REMARKS

This paper describes an extension of inverse Ailamujia distribution known as weighted inverse Ailamujia distribution has been established. Its several mathematical properties has been derived and discussed. The maximum likelihood procedure is used to obtain parameter estimation. Ultimately, the model's performance was validated employing two real data sets, and we can notice from tables 9.2 and 9.4 that the weighted inverse Ailamujia distribution (WIAD) has fewer values for the AIC, CAIC, and BIC statistics when contrasted to other models. As a conclusion, we demonstrate that the weighted inverse Ailamujia distribution delivers a superior match than the comparable ones.

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