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ORIGINAL ARTICLE

Weighted Inverse Log Logistic Probability Model: Statistical Properties and Applications

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Abstract: The concept of weighted distribution can be employed in development of proper model for life time data. In this paper a new weighted probability model called weighted inverse log logistic distribution is proposed. The statistical properties of weighted inverse log-logistic distribution are derived and the model parameters are estimated by maximum likelihood estimation. A simulation study is conducted to compare the different life time distribution. Finally, an application to real data set is presented for illustration.

Keywords: Weighted Distribution; Inverse Log Logistic; Maximum Likelihood Estimation.

1. Introduction

The log-logistic distribution is a derivative of the very popular logistic distribution which was initially developed to model population growth by Verhulst (1838). Since the development of logistic growth curve there have been several contributions suggesting alternative functional forms for growth whilst retaining the sigmoid and asymptotic property of the Verhulst logistic curve.In probability theory, the log-logistic distribution is a continuous probability distribution used in survival analysis as a parametric model whose probability density function (pdf) is given for events whose rate increases initially and decreases later, for example mortality rate from cancer following diagnosis or treatment. The inverse of log-logistic model also provides a greater flexibility in survival data sets. The probability density function (pdf) of the inverse log-logistic is given by

$$g(x;\alpha)=\frac{\alpha}{x^{(\alpha+1)}\left(1+x^{-\alpha}\right)^2}\ ;\ x>0,\alpha>0 \qquad (1)$$

The inverse log-logistic distribution is broadly used in practice and it is a substitute to the log-normal distribution since it presents a failure rate function that increases, touches a peak after some finite period and then declines gradually. The properties of the log-logistic distribution make it an attractive substitute to the log-normal and Weibull distributions in the analysis of reliability data (2003). Gupta *et al.*(1991) made a study of log-logistic model in survival analysis. Ragab *et al.* (1984) developed order statistics from the log-logistic distribution and their properties. Collet (2003) suggested the log-logistic distribution for modeling the time following a heart transplantation. Kantam *et al.* (2001) studied acceptance sampling based on life tests: log-logistic model.

2. Weighted Inverse Log-logistic distribution

Weighted distribution theory gives an integrated method to study with model design and data interpretation problems. Weighted distributions arise commonly in studies connected to reliability, survival analysis, analysis of family data, biomedicine, ecology and several other areas, see Stene (1981) and Oluyede and George (2002). Gupta and Tripathi (1996) studied the weighted version of the bivariate logarithmic series distribution, which has applications in many fields such as: ecology, social and behavioral sciences. Ahmed et al. (2016) discussed length biased weighted Lomax distribution with its applications. To existent the idea of a weighted distribution, suppose that X is a nonnegative random variable with its probability density function (pdf) $f(x)$, then the p.d.f. of the weighted random variable X_w is known by

$$f_w(x)=\frac{w(x)f(x)}{E(w(x))}\ x\geq 0 \qquad (2)$$

where $w(x)$ be a non-negative weight function.

Conditional upon the choice of the weight function $w(x)$, we have different weighted models. In this paper we have introduced a weighted family of inverse log-logistic distributions by taking the weight as x^c, to inverse log-logistic distribution. Substituting equation (1) and $w(x) = x^c$, in equation

(2) we get the pdf of weighted inverse of log-logistic distribution as given below

$$f(x;\alpha,c)=\frac{\alpha^2}{c\pi}\sin\left(\frac{\pi c}{\alpha}\right)\frac{x^{c-\alpha-1}}{\left(1+x^{-\alpha}\right)^2} \quad (3)$$

Figure 1.1 represents different shapes of probability density function of weighted inverse log logistic distribution.

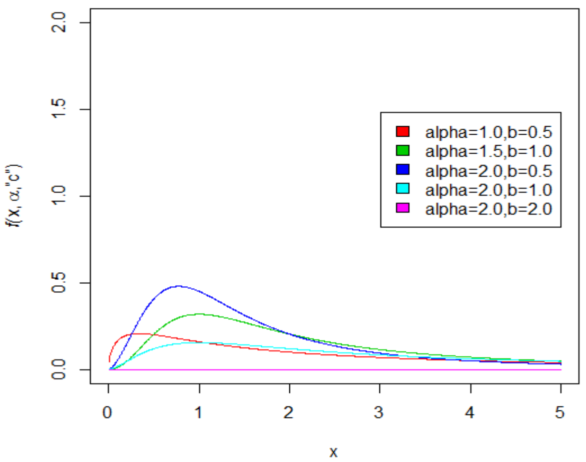

Figure 1.1: Pdf of Weighted Inverse log logistic Distribution

3. Statistical Properties

In this section we shall discuss structural properties of weighted inverse log-logistic distribution, especially moments, coefficient of variation, moment generating function.

3.1 Moments

Suppose X denote the random variable of weighted inverse log-logistic distribution with parameters α and c, then

$$E\left(X^r\right)=\mu_r^1=\int_0^\infty x^r f(x,\alpha,c)dx$$

$$=\frac{\alpha^2}{c\pi}\sin\left(\frac{\pi c}{\alpha}\right)\int_0^\infty\frac{x^{r+c-\alpha-1}}{\left(1+x^{-\alpha}\right)^2}dx \quad (4)$$

Substituting r = 1,2,3,4 in expression (4), we get first four moments

$$\text{Mean}=\mu_1'=\left(\frac{c+1}{c}\right)\frac{\sin\left(\frac{\pi c}{\alpha}\right)}{\sin\left[\left(\frac{c+1}{\alpha}\right)\pi\right]} \quad (5)$$

$$\mu_2'=\left(\frac{c+2}{c}\right)\frac{\sin\left(\frac{\pi c}{\alpha}\right)}{\sin\left[\left(\frac{c+2}{\alpha}\right)\pi\right]}$$

$$,\mu_3'=\left(\frac{c+3}{c}\right)\frac{\sin\left(\frac{\pi c}{\alpha}\right)}{\sin\left[\left(\frac{c+3}{\alpha}\right)\pi\right]},$$

$$\mu_4'=\left(\frac{c+4}{c}\right)\frac{\sin\left(\frac{\pi c}{\alpha}\right)}{\sin\left[\left(\frac{c+4}{\alpha}\right)\pi\right]}$$

$$\text{Variance}=\left(\frac{c+2}{c}\right)\frac{\sin\left(\frac{\pi c}{\alpha}\right)}{\sin\left[\left(\frac{c+2}{\alpha}\right)\pi\right]}-\left(\left(\frac{c+1}{c}\right)\frac{\sin\left(\frac{\pi c}{\alpha}\right)}{\sin\left[\left(\frac{c+1}{\alpha}\right)\pi\right]}\right)$$

Standard Deviation

$$\sigma=\left[\left(\frac{c+2}{c}\right)\frac{\sin\left(\frac{\pi c}{\alpha}\right)}{\sin\left[\left(\frac{c+2}{\alpha}\right)\pi\right]}-\left(\left(\frac{c+1}{c}\right)\frac{\sin\left(\frac{\pi c}{\alpha}\right)}{\sin\left[\left(\frac{c+1}{\alpha}\right)\pi\right]}\right)^2\right]^{\frac{1}{2}} \quad (6)$$

Coefficient of Variation

$$C.V=\frac{\sigma}{\mu_1'}=$$

$$\frac{\left[\left(\frac{c+2}{c}\right)\frac{\sin\left(\frac{\pi c}{\alpha}\right)}{\sin\left[\left(\frac{c+2}{\alpha}\right)\pi\right]}-\left(\left(\frac{c+1}{c}\right)\frac{\sin\left(\frac{\pi c}{\alpha}\right)}{\sin\left[\left(\frac{c+1}{\alpha}\right)\pi\right]}\right)^2\right]^{\frac{1}{2}}}{\left(\frac{c+1}{c}\right)\frac{\sin\left(\frac{\pi c}{\alpha}\right)}{\sin\left[\left(\frac{c+1}{\alpha}\right)\pi\right]}} \quad (7)$$

3.2 Moment generating function

In this sub section we derived the moment generating function of weighted inverse log-logistic distribution. From the definition of moment generating function we have

$$M_x(t)=E\left(e^{tx}\right)=\int_0^\infty e^{tx}f(x,\alpha,c)dx$$

$$=\int_0^\infty\sum_{j=0}^\infty\frac{(tx)^j}{j!}f(x,\alpha,c)dx$$

$$=\sum_{j=0}^\infty\frac{(t)^j}{j!}\int_0^\infty x^j f(x,\alpha,c)dx$$

$$=\sum_{j=0}^\infty\frac{(t)^j}{j!}\mu_j'$$

$$=\sum_{j=0}^\infty\frac{(t)^j}{j!}\left(\frac{c+j}{c}\right)\frac{\sin\left(\frac{\pi c}{\alpha}\right)}{\sin\left[\left(\frac{c+j}{\alpha}\right)\pi\right]} \quad (8)$$

3.3 Mode

In order to discuss monotonicity of weighted inverse log-logistic distribution, we take the logarithm of its pdf as follows

$$\log f(x,\alpha,c)=2\log\alpha-\log\pi c+\log\left(\sin\left(\frac{\pi c}{\alpha}\right)\right)+\log x^c \quad (9)$$

$$-2\log\left(1+x^{-\alpha}\right)$$

Differentiating the above equation with respect to x and equating to zero, we obtain

$$x = (-1)^{1/\alpha}\left(\frac{2\alpha + c}{c}\right)^{1/\alpha} \qquad (10)$$

4. Entropy

Entropy is a measure of variation of the uncertainty in the distribution of any random variable. It provides important tools to indicate variety in distributions at particular moments in time and to analyze evolutionary processes over time. In this section we derive the Shannon's and Reniy's Entropy for weighted inverse log logistic distribution.

4.1 Shannon's Entropy

Shannon Entropy is *H(x)* for weighted inverse log-logistic distribution is given by

$$H(x) = E[-\log f(x,\alpha,c)]$$

$$= \int_0^\infty -\log f(x,\alpha,c)dx$$

$$= \int_0^\infty \left(-\log\left(\frac{\alpha^2}{c\pi}\sin\left(\frac{\pi c}{\alpha}\right)\right) + \log x^c - 2\log\left(1 + x^{-\alpha}\right)\right)dx$$

$$H(x) = -\log\left(\frac{\alpha^2}{c\pi}\sin\left(\frac{\pi c}{\alpha}\right)\right) - E\left[\log x^c\right] + E\left[\log x^{\alpha+1}\right] \qquad (11)$$
$$+ E\left[\log\left(1 + x^{-\alpha}\right)^2\right]$$

4.2. Rényi Entropy

For a given probability distribution, Rényi (1961) gave an expression of the entropy function, so called Rényi entropy, defined by

$$Re(\gamma) = \frac{1}{1-\gamma}\log\left\{\int f^{\gamma}(x)dx\right\} \qquad (12)$$

Where γ > 0 and γ ≠ 0. For Weighted inverse log-logistic distribution in (3), we have

$$Re(\gamma) = \frac{1}{1-\gamma}\log\left\{\frac{\alpha^{2\gamma}}{(c\pi)^{\gamma}}\sin^{\gamma}\left(\frac{\pi c}{\alpha}\right)\int_0^\infty\left[\frac{x^{r+c-\alpha-1}}{\left(1+x^{-\alpha}\right)^2}\right]^{\gamma}dx\right\}$$

Evaluating the integral by suitable substitution by putting $x^{-\alpha} = t$ we get the following,

$$Re(\gamma) = \frac{1}{1-\gamma}\log\left\{\frac{\alpha^{2\gamma-1}}{(c\pi)^{\gamma}}\sin^{\gamma}\left(\frac{\pi c}{\alpha}\right)\int_0^\infty \frac{t^{\left(\frac{-c\gamma+\gamma-1}{\alpha}\right)+\gamma-1}}{(1+t)^2}dt\right\}$$

$$= \frac{1}{1-\gamma}\log\left\{\frac{\alpha^{2\gamma-1}}{(c\pi)^{\gamma}}\sin^{\gamma}\left(\frac{\pi c}{\alpha}\right)\Gamma\left(\frac{-c\gamma+\gamma-1}{\alpha}+\gamma\right)\Gamma\left(\frac{c\gamma-\gamma+1}{\alpha}+\gamma\right)\right\}$$

5. Estimation of parameter

In this section, we derive the estimates of parameters of weighted inverse log-logistic by various methods of estimation viz method of moments and maximum likelihood estimation.

5.1 Method of Maximum Likelihood Estimator

The method Maximum likelihood estimation is the most popular technique used for estimating the parameters of inverse log-logistic. Let $x_1, x_2, x_3, \ldots\ldots x_i$ be a random sample from the weighted inverse log-logistic, then the corresponding log likelihood function is given by

$$l(\theta) = \log L(x,\alpha,c) = \log\prod_{i-1}^{n} f(x_i,\alpha,c) = \sum_{i=1}^{n}\log f(x_i,\alpha,c)$$

$$\log f(x,\alpha,c) = 2\log\alpha - \log\pi c + \log\left(\sin\left(\frac{\pi c}{\alpha}\right)\right) + \log x^c \qquad (13)$$
$$-2\log\left(1 + x^{-\alpha}\right)$$

Now differentiating above with respect to the parameters, we obtain the normal equations

$$\frac{\partial \log L}{\partial\alpha} = 0 \Rightarrow \frac{2n}{\alpha} - \frac{n\pi c}{\alpha^2}\cot\left(\frac{\pi c}{\alpha}\right) + \sum_{i=1}^{n}\log x_i \qquad (14)$$
$$-\alpha\sum_{i=1}^{n}\frac{x_i^{-(\alpha+1)}}{\left(1 + x^{-\alpha}\right)} = 0$$

Consider the series expansion of

$$\cot\left(\frac{\pi c}{\alpha}\right) = \frac{1}{\left(\frac{\pi c}{\alpha}\right)} - \frac{\left(\frac{\pi c}{\alpha}\right)}{3} - \frac{\left(\frac{\pi c}{\alpha}\right)^3}{45} - \frac{\left(\frac{\pi c}{\alpha}\right)^5}{945} - \cdots \qquad (15)$$

By ratio test the series is convergence for $\left|\frac{c}{\alpha}\right| < 1$.

Using the series (15) by neglecting the second and higher degree terms and substituting in the expansion (14) we obtained the following normal equation,

$$\alpha^2\sum_{i=1}^{n}\frac{1}{x_i\left(1 + x_i^{\alpha}\right)} - \frac{\alpha^2}{2}\sum_{i=1}^{n}\log x_i + 2n\alpha = 0 \qquad (16)$$

This equation can be solved using Newton-Raphson method which is a powerful technique for solving equations iteratively and numerically.

$$\frac{1}{c} = \frac{\pi}{\alpha}\cot\left(\frac{\pi c}{\alpha}\right) + \frac{1}{n}\sum_{i=1}^{n}\log x_i \qquad (17)$$

Again by neglecting the third and higher degree terms in the expansion (15) and substituting in the expression (17) we get the following parameter $\hat{C}$,

$$\text{where, } \hat{c} = \frac{3\hat{\alpha}^2}{n\pi^2}\left[\sum_{i=1}^{n}\log x_i\right] \qquad (18)$$

5.2 Methods of Moments

Replacing sample moment with population moments, we get

$$\frac{1}{n}\sum_{i=1}^{n}x_i = \mu_1'$$

$$\Rightarrow \frac{1}{n}\sum_{i=1}^{n}x_i = \left(\frac{c+1}{c}\right)\frac{\sin\left(\frac{\pi c}{\alpha}\right)}{\sin\left[\left(\frac{c+1}{\alpha}\right)\pi\right]}$$

$$\overline{X}=\left(\frac{c+1}{c}\right)\frac{\sin\left(\frac{\pi c}{\alpha}\right)}{\sin\left[\left(\frac{c+1}{\alpha}\right)\pi\right]} \tag{19}$$

Now $\frac{1}{n}\sum_{i=1}^{n}x_i^2=\mu_2^{'}$

$$\Rightarrow \frac{1}{n}\sum_{i=1}^{n}x_i^2=\left(\frac{c+2}{c}\right)\frac{\sin\left(\frac{\pi c}{\alpha}\right)}{\sin\left[\left(\frac{c+2}{\alpha}\right)\pi\right]} \tag{20}$$

From the expressions (19) and (20) we get the following expression.

$$\frac{1}{n}\sum_{i=1}^{n}x_i^2=\left(\frac{c+2}{c+1}\right)\frac{\sin\left[\left(\frac{c+1}{\alpha}\right)\pi\right]}{\sin\left[\left(\frac{c+2}{\alpha}\right)\pi\right]} \tag{21}$$

Equations (19) and (21) are complex cannot be solved analytically, thus statistical R software can be used to solve these equations numerically. We can use iterative techniques to obtain the estimates.

6. Application

Data 1: The first data set represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal (1960). The data are as follows: 0.1, 0.33, 0.44, 0.56, 0.59, 0.59, 0.72, 0.74, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55, 2.54, 0.77

Data 11: The second data set correspond the failure times of 84 for a particular model aircraft windshield. This data are reported in the book "Weibull Models" by Murthy et al.(2004). This data consist of 84 failed windshield, the unit for measurement is 1000 h. The data are : 0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309,1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070,1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779,1.248, 2.010, 2.688, 3.924, 1.281,2.038, 2.823, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432,2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506,2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619,2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757,2.324, 3.376, 4.663.

In order to compare the two distribution models, we consider the criteria like AIC (Akaike information criterion, AICC (corrected Akaike information criterion) and BIC (Bayesian information criterion. The better distribution corresponds to lesser AIC, AICC and BIC values. From Table 6.1 and 6.2, it has been observed that the weighted inverse log logistic have the lesser AIC, AICC, -2logL and BIC values as compared to Inverse Log logistic Distribution. Hence we can conclude that weighted inverse log logistic distribution leads to a better fit as compared to inverse log logistic distribution.

Table 6.1: ML estimates and Criteria for Comparison for data of Survival times of a group pigs infected with virulent tubercle bacilli disease.

Distribution	Estimates	S.E	-2logL	AIC	AICC	BIC
Weighted Inverse log-logistic	$\alpha=2.97182$ c = 0.96035	0.325 84 0.261 65	196.0258	200.0258	200.1997	208.5545
Inverse log logistic	$\alpha=2.37185$	0.23255	217.8636	219.8636	219.9207	224.1967

Table 6.2: ML estimates and Criteria for Comparison for data correspond the failure times of 84 for a particular model aircraft windshield.

Distribution	Estimates	S.E	-2logL	AIC	AICC	BIC
Weighted Inverse log-logistic	$\alpha=2.51315$ c =1.2492	0.27381 0.25595	330.3822	334.3822	334.5303	343.3919
Inverse log logistic	$\alpha=1.57083$	0.13650	385.5481	387.5481	387.5968	391.9789

Conclusion

In this paper, we have introduced a new generalization of inverse log logistic distribution using the concept of weighting. The statistical properties of this distribution are derived and the model parameters are estimated by maximum likelihood estimation. Finally, an application to real data set is presented for illustration in engineering and medical sciences .The application of the weighted inverse log logistic distributions have also been demonstrated with real life examples from medical science. The results are compared with inverse log logistic distribution, revealed that the weighted inverse log logistic model provides a better fit than the inverse log logistic distribution.

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