Piece-wise Quasi-linear Approximation for Nonlinear Model Reduction

S. A. Nahvi, M. Nabi, and S. Janardhanan

***Abstract*—The trajectory piece-wise linear (TPWL) method is a popular technique for nonlinear model order reduction (MOR). Though widely studied, it has primarily been restricted to applications modeled by nonlinear systems with linear input operators. This paper is an effort to bridge this gap. We illustrate problems in the TPWL method in creating reduced order models for nonlinear systems with nonlinear input operators. We also propose a solution based on a quasi-linear formulation of the nonlinear system at approximation points. This results in a method for nonlinear MOR, called the trajectory piece-wise quasi-linear (TPWQ) method. TPWQ is formulated, numerically validated and a new technique to reduce the computational costs associated with simulating the quasi-linear systems is also demonstrated.**

***Index Terms*—Large dynamical systems, model order reduction, nonlinear systems, trajectory piecewise linear.**

I. Introduction

TPWL has been applied with success for nonlinear MOR in diverse application areas like computational fluid dynamics, power electronics and electromagnetics. However, these applications have generally been restricted to nonlinear systems with linear input operators. In this paper we study the efficacy of TPWL-ROMs in approximating dynamics of more general nonlinear systems.

Consider the class of nonlinear dynamical systems given by

$$\begin{aligned} \dot{x} &= F(x, u) \\ y &= Cx \end{aligned} \tag{1}$$

where $x \in R^n$ is the n dimensional state vector, $u \in R^k$ is the input, $y \in R^l$ is the output, $F : R^{(n+k)} \rightarrow R^{(n)}$ is the nonlinear vector field and C is $l \times n$ output matrix. To get the TPWL formulation corresponding to (1), $F(x, u)$ is linearized at (x_i, u_i)

$$F(x, u) = F(x_i, u_i) + A(x_i, u_i)(x - x_i) + B(x_i, u_i)(u - u_i). \tag{2}$$

Here $A(x_i, u_i)$ is the Jacobian of $F(x, u)$ with respect to x evaluated at (x_i, u_i) and $B(x_i, u_i)$ is the Jacobian of $F(x, u)$ with respect to u evaluated at (x_i, u_i). The linear system at (x_i, u_i) is hence

$$\begin{aligned} \dot{x} &= A(x_i, u_i)x + B(x_i, u_i)u + D(x_i, u_i) \\ y &= Cx \end{aligned} \tag{3}$$

where $D(x_i, u_i) = F(x_i, u_i) - A(x_i, u_i)x_i - B(x_i, u_i)u_i$.

Manuscript received April 30, 2013; revised June 25, 2013; accepted July 17, 2013. Date of current version November 18, 2013. This paper was recommended by Associate Editor A. Demir.

The authors are with the Department of Electrical Engineering, the Indian Institute of Technology Delhi, New Delhi 1100016, India (e-mail: s.a.nahvi@gmail.com; mnabi@ee.iitd.ac.in; janas@ee.iitd.ac.in).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TCAD.2013.2275260

Assuming the dominant dynamics to be restricted to an r dimensional subspace spanned by the columns of an orthonormal $n \times r$ matrix V, substituting $x = Vz$ in (3) and premultiplying by V^T we obtain the reduced submodel at x_i

$$\begin{aligned} \dot{z} &= V^T A(x_i, u_i)Vz + V^T B(x_i, u_i)u + V^T D(x_i, u_i)) \\ y &= CVz. \end{aligned} \tag{4}$$

The TPWL-ROM is expressed as a weighted sum of m such dimensionally reduced linear systems (4) and is given by

$$\begin{aligned} \dot{z} &= \textstyle\sum_{i=0}^{m-1} w_i(z, u)(A_{ir}z + B_{ir}u + D_{ir}) \\ y &= C_r z \end{aligned} \tag{5}$$

where $A_{ir} = V^T A(x_i, u_i)V$, $B_{ir} = V^T B(x_i, u_i)$, $D_{ir} = V^T D(x_i, u_i)$, $C_r = CV$. At a particular value of (z, u), the weight assignment strategy seeks to find the linear submodels formed at a state-input combination (z_i, u_i) that is closest to (z, u). Weights that are functions of both the state and input ((x, u) or (z, u)) have been used [1] and are decided by a distance measure given by

$$d_i = \gamma||z - z_i|| + \zeta||u - u_i|| \tag{6}$$

Here (γ, ζ) decide the relative importance associated with states and inputs and are heuristically chosen. It is obvious that two different criteria are being balanced to switch on the right submodels (unlike the case of input-affine nonlinear systems as in [2]) Amongst the two, the distance in inputs, $||u - u_i||$, is unrelated to the nonlinear system being approximated. This criteria may lead to a reduction in the weight given to the most relevant submodel with a sharp change in the input, as would be illustrated in Section II-A. The problem increases when training and evaluation inputs are different, as a more pronounced difference in u is likely to skew the weights in the wrong direction. Further no method to select (γ, ζ) and decide the relative importance of the two criteria is available. The problem can be solved by retaining the nonlinearity of the input operator, and assigning weights on the basis of the distance in states alone. In the following paragraphs it is shown how this can be done.

II. TPWQ Formulation

It is proposed that the system (1) is linearized with respect to the states alone, this quasi-linearization at x_i yields

$$F(x, u) = F(x_i, u) + A(x_i, u)(x - x_i) \tag{7}$$

where $A(x_i, u)$ is the Jacobian of $F(x, u)$ with respect to x evaluated at x_i. Hence the quasi-linear system at x_i would be given by

$$\begin{aligned} \dot{x} &= A(x_i, u)x + B(x_i, u) \\ y &= Cx \end{aligned} \tag{8}$$

where $B(x_i, u) = F(x_i, u) - A(x_i, u)x_i$. Assuming that the dynamics can be restricted to the subspace spanned by V and premultiplying by V^T gives the sub-model at each x_i

$$\begin{aligned} \dot{z} &= V^T A(x_i, u)Vz + V^T B(x_i, u) \\ y &= CVz. \end{aligned} \tag{9}$$