Chapter

Scaled Ambiguity Function Associated with Quadratic-Phase Fourier Transform

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Abstract

Quadratic-phase Fourier transform (QPFT) as a general integral transform has been considered into Wigner distribution (WD) and Ambiguity function (AF) to show more powerful ability for non-stationary signal processing. In this article, a new version of ambiguity function (AF) coined as scaled ambiguity function associated with the Quadratic-phase Fourier transform (QPFT) is proposed. This new version of AF is defined based on the QPFT and the fractional instantaneous auto-correlation. Firstly, we define the scaled ambiguity function associated with the QPFT (SAFQ). Then, the main properties including the conjugate-symmetry, shifting, scaling, marginal and Moyal's formulae of SAFQ are investigated in detail, the results show that SAFQ can be viewed as the generalization of the classical AF. Finally, the newly defined SAFQ is used for the detection of linear-frequency-modulated (LFM) signals.

Keywords: ambiguity function, quadratic-phase Fourier transform, Moyal's formula, modulation, linear frequency-modulated signal

1. Introduction

The Fourier transform is indeed an indispensable tool for the time-frequency analysis of the stationary signals. Due to its success stories FT has profoundly influenced the mathematical, biological, chemical and engineering communities over decades, but FT can not analyze non-stationary signals as it can not provide any valid information despite the localization properties of the spectral contents. FT only allows us to visualize the signals either in time or frequency domain, but not in both domains simultaneously. In Refs. [1–3], Castro et al. introduced a superlative generalized version of the Fourier transform(FT) called quadratic-phase Fourier transform(QPFT), which not only treats uniquely both the transient and non-transient signals in a nice fashion but also with non-orthogonal directions. The QPFT is actually a generalization of several well known transforms like Fourier, fractional Fourier and linear canonical transforms, offset linear canonical transform whose kernel is in the exponential form.

Many researches have been carried on quadratic-phase Fourier transform(see [4, 5]). With the fact that the QPFT is monitored by a bunch of free parameters, it has evolved as an effective tool for the representation of signals. A notable consideration has been given in the extension of the Wigner distributions to the classical QPFT and its generalizations. More can be found in Refs. [6–9].

On the other hand, the classical ambiguity function (AF) and Wigner distribution (WD) are the basic parametric time-frequency analysis tools, evolved for the analysis of time-frequency characteristics of non-stationary signals [10–14]. At the same time, the linear frequency-modulated (LFM) signal, a typical non-stationary signal, is widely used in communications, radar and sonar system. Many algorithms and methods have been proposed in view of LFM. The most important among them are the AF and WD [10, 13, 15–19], defined as the Fourier transform of the classical instantaneous autocorrelation function $\omega\left(t+\frac{\tau}{2}\right)\omega^*\left(t-\frac{\tau}{2}\right)$ for t and τ, (superscript $*$ denotes complex conjugate) respectively. It is well known that the AF offers perfect localization (localized on a straight line) to the mono-component LFM signals but cross terms appear while dealing with multi-component LFM signals as they are quadratic in nature. However these cross terms become troublesome if the frequency rate of one component approaches other. This drawback of AF gave rise to a series of different classes of time- frequency representation tools (see [20–27]). In Ref. [28], authors used fractional instantaneous auto-correlation $\omega\left(t+k\frac{\tau}{2}\right)\omega^*\left(t-k\frac{\tau}{2}\right)$ found in the definition of fractional bi-spectrum [29], which is parameterized by a constant $k\in\mathbb{Q}^+$ to introduced a scaled version of the conventional WD. Later Dar and Bhat [30] introduced the scaled version of Ambiguity function and Wigner distribution in the linear canonical transform domain. They also introduced scaled version of Wigner distribution in the offset linear canonical transform [31–35], hence provides a novel way for the improvement of the cross-term reduction time–frequency resolution and angle resolution.

Keeping in mind the degree of freedom corresponding to the choice of a factor k in the fractional instantaneous auto-correlation and the extra degree of freedom present in QPFT, we introduce a novel scaled ambiguity function in the quadratic-phase Fourier transform domain (SAFQ), which gives a unique treatment for all classical classes of AF's. Hence, it is good to study rigorously the SAFQ which will be effective for signal processing theory and applications especially for detection and estimation of LFM signals.

1.1 Paper contributions

The contributions of this paper are summarized below:

- To introduce a scaled ambiguity function associated with the quadratic-phase Fourier transform.

- To study the fundamental properties of the SAFQ, including the conjugate symmetry, time marginal, non-linearity, time shift, frequency shift, frequency marginal, scaling and Moyal formula.

- To show the of advantage of the theory, we provide the applications of the proposed distribution in the detection of single-component and bi-component linear-frequency-modulated (LFM) signal.

1.2 Paper outlines

The paper is organized as follows: In Section 2, we gave a brief review of QPFT and introduce AF associated with it. The definition and the properties of the SAFQ are studied in Section 3. In Section 4, the applications of the proposed distribution for the detection of single-component and bi-component LMF signals is provided. Finally, a conclusion is drawn in Section 5.

2. Preliminary

In this section, we gave the definitions of the Quadratic-phase Fourier transform (QPFT), the ambiguity function associated with QPFT and the scaled ambiguity function which will be needed throughout the paper.

2.1 Quadratic-phase Fourier transform (QPFT)

For a given set of parameters of $\Omega = (A, B, C, D, E)$,$B \neq 0$ the quadratic-phase Fourier transform any signal $\omega(t)$ is defined by [1–3]

$$\mathcal{Q}^{\Omega}[\omega](u) = \int_{\mathbb{R}} \omega(t) K_{\Omega}(t, u) dt, \tag{1}$$

where the quadratic-phase Fourier kernel $K_{\Omega}(t, w)$ is given by

$$K_{\Omega}(t, u) = \sqrt{\frac{B}{2\pi i}} e^{\left(At^2+Btu+Cu^2+Dt+Eu\right)}, \quad A,B,C,D.E \in \mathbb{R}. \tag{2}$$

2.2 Ambiguity function in the quadratic-phase fourier domain (AFQ)

Authors in Refs. [7, 8] defined the AF associated with the LCT, using the same procedure we can define the AF associated with QPFT (AFQ) as

$$AFQ^{\Omega}_{\omega(t)}(\tau, u) = \int_{\mathbb{R}} \omega\left(t + \frac{\tau}{2}\right) \omega^* \left(t - \frac{\tau}{2}\right) K_{\Omega}(\tau, u) dt, \tag{3}$$

2.3 Scaled ambiguity function

For a finite energy signal the scaled Ambiguity function (SAF) is defined as Ref. [30].

$$SAF_{\omega(t)}(\tau, u) = \int_{\mathbb{R}} \omega\left(t + k\frac{\tau}{2}\right) \omega^* \left(t - \frac{\tau}{2}\right) e^{-iut} dt, \tag{4}$$

where $k \in \mathbb{Q}^+$ the set of positive rational numbers.

3. Scaled ambiguity function associated with quadratic-phase fourier transform (SAFQ)

In this section, we shall introduce the notion of the scaled Ambiguity function associated with QPFT followed by some of its basic properties.

3.1 Definition of the scaled AFQ

Thanks to the scaled AF, we obtain obtain different expressions for the SAFQ as follows:

$$\begin{aligned} SAF_{\omega(t)}(\tau, u) &= \int_{\mathbb{R}} \omega\left(t + k\frac{\tau}{2}\right)\omega^*\left(t - k\frac{\tau}{2}\right)e^{-iut}dt \\ &= \int_{\mathbb{R}} \omega\left(t + k\frac{\tau}{2}\right)e^{-i\frac{u}{2}\left(t+k\frac{\tau}{2}\right)}\omega^*\left(t - \frac{\tau}{2}\right)e^{-i\frac{u}{2}\left(t-k\frac{\tau}{2}\right)}dt \\ &= \int_{\mathbb{R}} \overline{\omega}_u\left(t + k\frac{\tau}{2}\right)\hat{\omega}_u^*\left(t - k\frac{\tau}{2}\right)dt, \end{aligned} \tag{5}$$

where

$$\overline{\omega}_u(t) = \omega(t)e^{-i\frac{u}{2}t} \quad \text{and} \quad \hat{\omega}_u(t) = \omega(t)e^{i\frac{u}{2}t}. \tag{6}$$

On replacing the Fourier kernel in (6) with the QPFT kernel, we obtain

$$\overline{\omega}_u^{\Omega}(t) = \omega(t)K_{\Omega}\left(t, \frac{u}{2}\right) \quad \text{and} \quad \hat{\omega}_u^{\Omega}(t) = \omega(t)K_{\Omega}\left(t, -\frac{u}{2}\right). \tag{7}$$

Thus, we obtain a new version of scaled AF associated with the QPFT by replacing $\overline{\omega}_u(t)$ with $\overline{\omega}_u^{\Omega}(t)$ and $\hat{\omega}_u(t)$ with $\hat{\omega}_u^{\Omega}(t)$ in (5), i.e.,

$$\begin{aligned} SAF^{\Omega}_{\omega(t)}(\tau, u) &= \int_{\mathbb{R}} \overline{\omega}_u^{\Omega}\left(t + k\frac{\tau}{2}\right)\hat{x}_u^{\Omega\,*}\left(t - k\frac{\tau}{2}\right)dt \\ &= \int_{\mathbb{R}} \omega\left(t + k\frac{\tau}{2}\right)K_{\Omega}\left(t + k\frac{\tau}{2}, \frac{u}{2}\right)\omega^*\left(t - k\frac{\tau}{2}\right)K_{\Omega}^*\left(t - k\frac{\tau}{2}, \frac{-u}{2}\right)dt \\ &= \frac{B}{2\pi}\int_{\mathbb{R}} \omega\left(t + k\frac{\tau}{2}\right)\omega^*\left(t - k\frac{\tau}{2}\right)e^{i[(2Ak\tau+Bu)t+Dk\tau+Eu]}dt. \end{aligned} \tag{8}$$

With the virtue of above equation we have following definition.

Definition 3.1. *The scaled Ambiguity function associated with quadratic-phase Fourier transform of a signal '$\omega(t)$' in $L^2(\mathbb{R})$ with respect the real parameter set $\Omega = (A, B, C, D, E)$,$B \neq 0$ is defined as*

$$SAF^{\Omega}_{\omega(t)}(\tau, u) = \frac{B}{2\pi}\int_{\mathbb{R}} \omega\left(t + k\frac{\tau}{2}\right)\omega^*\left(t - k\frac{\tau}{2}\right)e^{i[(2Ak\tau+Bu)t+Dk\tau+Eu]}dt, \tag{9}$$

where $k \in \mathbb{Q}^+$.

It is worth to mention that if we change the parameter $\Omega = (A, B, C, D, E)$ in the Definition 3.1, we have the following important deductions:

i. When the parameter $\Omega = (A/2B, -1/B, C/2B, 0, 0)$ is chosen and multiplying the right side of (9) by -1, the SAFQ (9) yields the scaled ambiguity function associated with linear canonical transform [30]:

$$SAF^{\Omega}_{\omega(t)}(\tau, u) = \frac{1}{2\pi B}\int_{\mathbb{R}} \omega\left(t + k\frac{\tau}{2}\right)\omega^*\left(t - k\frac{\tau}{2}\right)e^{i\frac{1}{B}(Ak\tau - u)t}dt. \tag{10}$$

ii. For the set $\Omega = (\cot\zeta/2, -\csc\zeta, \cot\zeta/2, 0, 0), \zeta \neq 2\pi$ and multiplying the right side of (9) by -1 the SAFQ (9) yields the novel scaled AF associated with fractional Fourier transform:

$$SAF^{\zeta}_{\omega(t)}(t, u) = \frac{1}{2\pi \sin\zeta}\int_{\mathbb{R}} \omega\left(t + k\frac{\theta}{2}\right)\omega^*\left(t - k\frac{\tau}{2}\right)e^{i((k\cot\zeta\tau - u\csc\zeta)t}dt. \tag{11}$$

iii. When the parameter is choosen as $\Omega = (0,1,0,0,0)$ is chosen, the scaled AFQ (4) boils down to the classical scaled AF given in Ref. [30]. In addition of above if we take $k = 1$, it reduce to classical Amniguity function.

3.2 Properties of the scaled AFOL

In this subsection, we investigate some general properties of the scaled AFQ with their detailed proofs. These properties play vital role in signal representation. We shall see the differences between the scaled versions and conventional ones.

Property 3.1 (symmetry property) *For $\omega(t) \in L^2(\mathbb{R})$, then scaled AFOL of the signals $\omega^*(t)$ and $P[\omega(t)]$ have the following forms*

$$SAF^{\Omega}_{\omega(t)^*}(\tau, u) = SAF^{\Omega'}_{\omega(t)}(-\tau, -u) \tag{12}$$

where $\Omega' = (-A. - B, C, -D, -E)$.
and

$$SAF^{\Omega}_{P[\omega(t)]}(\tau, u) = -SAF^{\overline{\Omega}}_{\omega(t)}(-\tau, -u), \tag{13}$$

where $P[\omega(t)] = \omega(-t)$ and $\overline{\Omega} = (A, B, C, -D, -E)$.
Proof. From Definition 3.1, we have

$$\begin{aligned}
&SAF^{\Omega}_{\omega(t)^*}(\tau, u)\\
&= \frac{B}{2\pi}\int_{\mathbb{R}} \omega^*\left(t + k\frac{\tau}{2}\right)\omega\left(t - k\frac{\tau}{2}\right)e^{i[(2Ak\tau + Bu)t + Dk\tau + Eu]}dt\\
&= \frac{B}{2\pi}\int_{\mathbb{R}} \omega\left(t + k\frac{(-\tau)}{2}\right)\omega^*\left(t - k\frac{(-\tau)}{2}\right)e^{i[(2Ak\tau + Bu)t + Dk\tau + Eu]}dt\\
&= \frac{B}{2\pi}\int_{\mathbb{R}} \omega\left(t + k\frac{(-\tau)}{2}\right)\omega^*\left(t - k\frac{(-\tau)}{2}\right) \times e^{i[\{2(-A)k(-\tau) + (-B)(-u)\}t + (-D)k(-\tau) + (-E)(-u)]}dt\\
&= SAF^{\Omega'}_{\omega(t)}(-\tau, -u), \quad \textit{where} \quad \Omega' = (-A. - B, C, -D, -E).
\end{aligned}$$

which prove (12).
Now, we move forward to prove (13)
From (9), we have

$$\begin{aligned}
&SAF^{\Omega}_{P[\omega(t)]}(\tau, u)\\
&= \frac{B}{2\pi}\int_{\mathbb{R}} P\omega\left(t + k\frac{\tau}{2}\right)P\omega^*\left(t - k\frac{\tau}{2}\right)e^{i[(2Ak\tau+Bu)t+Dk\tau+Eu]}dt\\
&= \frac{B}{2\pi}\int_{\mathbb{R}} \omega\left(-t - k\frac{\tau}{2}\right)\omega^*\left(-t + k\frac{\tau}{2}\right)e^{i[(2Ak\tau+Bu)t+Dk\tau+Eu]}dt\\
&= \frac{B}{2\pi}\int_{\mathbb{R}} \omega\left(-t - k\frac{\tau}{2}\right)\omega^*\left(-t + k\frac{\tau}{2}\right)e^{i[\{2Ak(-\tau)+B(-u)\}(-t)+(-D)k(-\tau)+(-E)(-u)]}dt\\
&= -\frac{B}{2\pi}\int_{\mathbb{R}} \omega\left(v + k\frac{-\tau}{2}\right)\omega^*\left(v - k\frac{-\tau}{2}\right)e^{i[\{2Ak(-\tau)+B(-u)\}v+(-D)k(-\tau)+(-E)(-u)]}dv\\
&= -SAF^{\overline{\Omega}}_{\omega(t)}(-\tau, -u), \quad \overline{\Omega} = (A, B, C, -D, -E).
\end{aligned}$$

which completes the proof. □

Property 3.2 (Time shift). *The SAFQ of a signal* $\omega(t-\lambda)$ *can be expressed as:*

$$SAF^{\Omega}_{\omega(t-\lambda)}(\tau, u) = e^{i\lambda(2Ak\tau+Bu)}SAF^{\Omega}_{\omega(t)}(\tau, u). \tag{14}$$

Proof. From (9), we obtain

$$SAF^{\Omega}_{\omega(t-\lambda)}(\tau, u) = \frac{B}{2\pi}\int_{\mathbb{R}} \omega\left(t - \lambda + k\frac{\tau}{2}\right)\omega^*\left(t - \lambda - k\frac{\tau}{2}\right)e^{i[(2Ak\tau+Bu)t+Dk\tau+Eu]}dt.$$

Setting $t - \lambda = s$, we have from last equation

$$\begin{aligned}
SAF^{\Omega}_{\omega(t-\lambda)}(\tau, u) &= \frac{B}{2\pi}\int_{\mathbb{R}} \omega\left(s + k\frac{\tau}{2}\right)\omega^*\left(s - k\frac{\tau}{2}\right)e^{i[(2Ak\tau+Bu)s+Dk\tau+Eu]}ds\\
&= e^{i\lambda(2Ak\tau+Bu)}\frac{B}{2\pi}\int_{\mathbb{R}} \omega\left(s + k\frac{\tau}{2}\right)\omega^*\left(s - k\frac{\tau}{2}\right)e^{\frac{i}{b}[(ak\tau-u)s+ku_0\tau-u(du_0-bw_0)]}ds\\
&= e^{i\lambda(2Ak\tau+Bu)}SAF^{\Omega}_{\omega(t)}(\tau, u).
\end{aligned}$$

Which completes the proof of (14). □

Property 3.3 (Frequency shift). *The SAFQ of a signal* $\omega(t)e^{ivt}$ *can be expressed as:*

$$SAF^{\Omega}_{\omega(t)e^{ivt}}(\tau, u) = e^{ivk\tau}SAF^{\Omega}_{\omega(t)}(\tau, u) \tag{15}$$

Proof. From (9), we have

$$\begin{aligned}
SAF^{\Omega}_{\omega(t)e^{ivt}}(\tau, u) &= \frac{B}{2\pi}\int_{\mathbb{R}} \omega\left(t + k\frac{\tau}{2}\right)e^{iv\left(t+k\frac{\tau}{2}\right)}\omega^*\left(t - \frac{\tau}{2}\right)e^{-iv\left(t-k\frac{\tau}{2}\right)}\\
&\qquad \times e^{i[(2Ak\tau+Bu)t+Dk\tau+Eu]}dt\\
&= \frac{B}{2\pi}\int_{\mathbb{R}} \omega\left(t + k\frac{\tau}{2}\right)\omega^*\left(t - \frac{\tau}{2}\right)e^{ivk\tau}\\
&\qquad \times e^{i[(2Ak\tau+Bu)t+Dk\tau+Eu]}dt\\
&= e^{ivk\tau}\frac{B}{2\pi}\int_{\mathbb{R}} \omega\left(t + k\frac{\tau}{2}\right)\omega^*\left(t - \frac{\tau}{2}\right)\\
&\qquad \times e^{i[(2Ak\tau+Bu)t+Dk\tau+Eu]}dt\\
&= e^{ivk\tau}SAF^{\Omega}_{\omega(t)}(\tau, u).
\end{aligned}$$

Which completes the proof □

Property 3.4 (Non-linearity). *Let $\omega(t) = \omega_1(t) + \omega_2(t)$ be in $L^2(\mathbb{R})$, then we have*

$$SAF^{\Omega}_{\omega(t)}(\tau, u) = SAF^{\Omega}_{\omega_1(t)}(\tau, u) + SAF^{\Omega}_{\omega_2(t)}(\tau, u) + SAF^{\Omega}_{\omega_1,\omega_2}(\tau, u) + SAF^{\Omega}_{\omega_2,\omega_1}(\tau, u) \tag{16}$$

Proof. From Definition 3.1, we have

$$\begin{aligned}
&SAF^{\Omega}_{\omega(t)}(\tau, u)\\
&= \frac{B}{2\pi}\int_{\mathbb{R}} (\omega_1 + \omega_2)\left(t + k\frac{\tau}{2}\right)(\omega_1 + \omega_2)^*\left(t - k\frac{\tau}{2}\right)e^{i[(2Ak\tau+Bu)t+Dk\tau+Eu]}dt\\
&= \frac{B}{2\pi}\int_{\mathbb{R}} \Big[\left(\omega_1\left(t + k\frac{\tau}{2}\right) + \omega_2\left(t + k\frac{\tau}{2}\right)\right)\\
&\qquad \left(\omega_1{}^*\left(t - k\frac{\tau}{2}\right) + \omega_2{}^*\left(t - k\frac{\tau}{2}\right)\right)\Big]e^{i[(2Ak\tau+Bu)t+Dk\tau+Eu]}dt\\
&= \frac{B}{2\pi}\int_{\mathbb{R}} \Big[\omega_1\left(t + k\frac{\tau}{2}\right)\omega_1{}^*\left(t - k\frac{\tau}{2}\right) + \omega_2\left(t + k\frac{\tau}{2}\right)\omega_2{}^*\left(t - k\frac{\tau}{2}\right)\\
&\qquad +\omega_1\left(t + k\frac{\tau}{2}\right)\omega_2{}^*\left(t - k\frac{\tau}{2}\right) + \omega_2\left(t + k\frac{\tau}{2}\right)\omega_1{}^*\left(t - k\frac{\tau}{2}\right)\Big]\\
&\qquad \times e^{i[(2Ak\tau+Bu)t+Dk\tau+Eu]}dt\\
&= SAF^{\Omega}_{\omega_1}(\tau, u) + SAF^{\Omega}_{\omega_2}(\tau, u) + SAF^{\Omega}_{\omega_1,\omega_2}(\tau, u) + SAF^{\Omega}_{\omega_2,\omega_1}(\tau, u).
\end{aligned}$$

Thus completes the proof. □

Property 3.5 (Frequency marginal property). *The frequency marginal property of SAFQ is given by*

$$\int_{\mathbb{R}} SAF^{\Omega}_{\omega(t)}(\tau, u)d\tau = \frac{1}{k}\mathcal{Q}^{\Omega}[\omega(t)]\left(\frac{u}{2}\right)\mathcal{Q}^{*\Omega}[\omega(t)]\left(\frac{-u}{2}\right) \tag{17}$$

Proof. From Definition 3.1, we have

$$\int_{\mathbb{R}} SAF^{\Omega}_{\omega(t)}(\tau, u)d\tau = \frac{B}{2\pi}\int_{\mathbb{R}^2} \omega\left(t + k\frac{\tau}{2}\right)\omega^*\left(t - \frac{\tau}{2}\right)e^{i[(2Ak\tau+Bu)t+Dk\tau+Eu]}dtd\tau.$$

Making change of variable $t + k\frac{\tau}{2} = s$, above equation yields

$$\int_{\mathbb{R}} SAF^{\Omega}_{\omega(t)}(\tau, u)d\tau = \frac{B}{\pi k}\int_{\mathbb{R}^2} \omega(s)\omega^*(2t - s)e^{i[\{4A(s-t)+Bu\}t+2D(s-t)+Eu]}dsdt.$$

Now setting $2t = s + v$, we get

$$
\begin{aligned}
&\int_{\mathbb{R}} SAF^{\Omega}_{\omega(t)}(\tau, u)d\tau \\
&= \frac{B}{2\pi k}\int_{\mathbb{R}^2} \omega(s)\omega^*(v)e^{i\left[\left\{4A\left(s-\frac{s+v}{2}\right)-Bu\right\}\left(\frac{s+v}{2}\right)+2D\left(s-\frac{s+v}{2}\right)+Eu\right]}dsdv \\
&= \frac{B}{2\pi k}\int_{\mathbb{R}^2} \omega(s)\omega^*(v)e^{i\left[\{2A(s-v)+Bu\}\left(\frac{s+v}{2}\right)+D(s-v)+Eu\right]}dsdv \\
&= \frac{B}{2\pi k}\int_{\mathbb{R}^2} \omega(s)\omega^*(v)e^{i\left[A\left(s^2-v^2\right)+Bu\left(\frac{s+v}{2}\right)+D(s-v)+Eu\right]}dsdv \\
&= \frac{B}{2\pi k}\int_{\mathbb{R}} \omega(s)e^{i\left[As^2+Bs\left(\frac{u}{2}\right)+C\left(\frac{u}{2}\right)^2+Ds+E\left(\frac{u}{2}\right)\right]}ds \\
&\qquad \times\int_{\mathbb{R}} \omega^*(v)e^{-i\left[Av^2+Bv\left(\frac{-u}{2}\right)+C\left(\frac{-u}{2}\right)^2+Dv+E\left(\frac{-u}{2}\right)\right]}dv \\
&= \frac{1}{k}\int_{\mathbb{R}} \omega(s)\sqrt{\frac{B}{2i\pi}}e^{i\left[As^2+Bs\left(\frac{u}{2}\right)+C\left(\frac{u}{2}\right)^2+Ds+E\left(\frac{u}{2}\right)\right]}ds \\
&\qquad \times\left[\int_{\mathbb{R}} \omega(v)\sqrt{\frac{B}{2i\pi}}e^{i\left[Av^2+Bv\left(\frac{-u}{2}\right)+C\left(\frac{-u}{2}\right)^2+Dv+E\left(\frac{-u}{2}\right)\right]}dv\right]^* \\
&= \frac{1}{k}\int_{\mathbb{R}} \omega(s)K_{\Omega}\left(s, \frac{u}{2}\right)ds\left[\int_{\mathbb{R}} \omega(v)K_{\Omega}\left(v, \frac{-u}{2}\right)dv\right]^* \\
&= \frac{1}{k}\mathcal{Q}^{\Omega}[\omega(t)]\left(\frac{u}{2}\right)\mathcal{Q}^{*\,\Omega}[\omega(t)]\left(\frac{-u}{2}\right).
\end{aligned}
$$

Which completes the proof. □

Property 3.6 (Scaling property). *For a signal $\tilde{\omega}(t) = \sqrt{\sigma}\omega(\sigma t)$ the SAFQ has the following form:*

$$
SAF^{\Omega}_{\tilde{\omega}(t)}(\tau, u) = SAF^{\Omega'}_{\omega(t)}\left(\sigma\tau, \frac{u}{\sigma}\right), \tag{18}
$$

where $\Omega' = \left(\frac{A}{\sigma^2}, B, C, \frac{D}{\sigma}, \sigma E\right)$.

Proof. From (9), we have

$$
SAF^{\Omega}_{\tilde{\omega}(t)}(\tau, u) = \frac{\sigma B}{2\pi}\int_{\mathbb{R}} \omega\left(\sigma t + \sigma k\frac{\tau}{2}\right)\omega^*\left(\sigma t - \sigma k\frac{\tau}{2}\right)e^{i[(2Ak\tau+Bu)t+Dk\tau+Eu]}dt.
$$

Setting $\sigma t = \eta$, above equation yields

$$
\begin{aligned}
&SAF^{\Omega}_{\tilde{\omega}(t)}(\tau, u) \\
&= \frac{\sigma B}{2\pi}\int_{\mathbb{R}} \omega\left(\sigma t + \sigma k\frac{\tau}{2}\right)\omega^*\left(\sigma t - \sigma k\frac{\tau}{2}\right)e^{i\left[(2Ak\tau+Bu)\frac{\eta}{\sigma}+Dk\tau+Eu\right]}.\frac{d\eta}{\sigma} \\
&= \frac{\sigma B}{2\pi}\int_{\mathbb{R}} \omega\left(\sigma t + \sigma k\frac{\tau}{2}\right)\omega^*\left(\sigma t - \sigma k\frac{\tau}{2}\right)e^{i\left[\left(2\frac{A}{\sigma^2}k(\sigma\tau)+B\frac{u}{\sigma}\right)\eta+Dk\tau+Eu\right]}.\frac{d\eta}{\sigma} \\
&= \frac{B}{2\pi}\int_{\mathbb{R}} \omega\left(\sigma t + \sigma k\frac{\tau}{2}\right)\omega^*\left(\sigma t - \sigma k\frac{\tau}{2}\right)e^{i\left[\left(2\frac{A}{\sigma^2}k(\sigma\tau)+B\left(\frac{u}{\sigma}\right)\right)\eta+\frac{D}{\sigma}k(\sigma\tau)+\sigma E\left(\frac{u}{\sigma}\right)\right]}.d\eta \\
&= SAF^{\Omega'}_{\omega(t)}\left(\sigma\tau, \frac{u}{\sigma}\right),
\end{aligned}
$$

where $\Omega' = \left(\frac{A}{\sigma^2}, B, C, \frac{D}{\sigma}, \sigma E\right)$.

This proves (18). □

Property 3.7 (Moyal formula). *The Moyal formula of the SAFQ has the following form:*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} SAF^{\Omega}_{\omega_1(t)}(\tau, u)\left[SAF^{\Omega}_{\omega_2(t)}(\tau,\ u)\right]^* d\tau du = \frac{B}{2\pi k}|\langle\omega_1(t), \omega_2(t)\rangle|^2. \tag{19}$$

Proof. From (9), we have

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} SAF^{\Omega}_{\omega_1(t)}(t, u)\left[SAF^{\Omega}_{\omega_2(t)}(t,\ u)\right]^* dtdu\\
&= \left(\frac{B}{2\pi}\right)^2\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \omega_1\left(t+k\frac{\tau}{2}\right)\omega_1^*\left(t-k\frac{\tau}{2}\right)\omega_2^*\left(t'+k\frac{\tau}{2}\right)\omega_2\left(t'-k\frac{\tau}{2}\right)\\
&\qquad \times e^{i[(2Ak\tau+Bu)t+Dk\tau+Eu]}e^{-i[(2Ak\tau+Bu)t'+Dk\tau+Eu]}d\tau dt' dtdu\\
&= \left(\frac{B}{2\pi}\right)^2\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \omega_1\left(t+k\frac{\tau}{2}\right)\omega_1^*\left(t-\frac{\tau}{2}\right)\omega_2^*\left(t+k\frac{\tau}{2}\right)\omega_2\left(t-k\frac{\tau}{2}\right)\\
&\qquad \times e^{i(2Ak\tau+Bu)(t-t')}d\tau dudtdt'\\
&= \frac{B}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \omega_1\left(t+k\frac{\tau}{2}\right)\omega_1^*\left(t-\frac{\tau}{2}\right)\omega_2^*\left(t+k\frac{\tau}{2}\right)\omega_2\left(t-k\frac{\tau}{2}\right)\\
&\qquad \times e^{i2Ak\tau(t-t')}\left(\frac{B}{2\pi}\int_{\mathbb{R}} e^{iBu(t-t')}du\right)d\tau dtdt'\\
&= \frac{B}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \omega_1\left(t+k\frac{\tau}{2}\right)\omega_1^*\left(t-\frac{\tau}{2}\right)\omega_2^*\left(t+k\frac{\tau}{2}\right)\omega_2\left(t-k\frac{\tau}{2}\right)\\
&\qquad \times e^{i2Ak\tau(t-t')}\delta(t-t')dt'd\tau dt\\
&= \frac{B}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \omega_1\left(t+k\frac{\tau}{2}\right)\omega_1^*\left(t-\frac{\tau}{2}\right)\omega_2^*\left(t+k\frac{\tau}{2}\right)\omega_2\left(t-k\frac{\tau}{2}\right)d\tau dt
\end{aligned}$$

By making the change of variable $s = t + k\frac{\tau}{2}$, we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \mathcal{W}^{A,k}_{\omega_1(t)}(t, u)\left[\mathcal{W}^{A,k}_{\omega_2(t)}(t,\ u)\right]^* d\tau du = \frac{B}{k\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \omega_1(s)\omega_1^*(2t-s)\omega_2^*(s)\omega_2(2t-s)dsdt$$

Now taking $2t - s = v$, we obtain

$$\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}} \mathcal{W}^{A,k}_{\omega_1}(t, u)\left[\mathcal{W}^{A,k}_{\omega_2}(t,\ u)\right]^* d\tau du &= \frac{B}{2\pi k}\int_{\mathbb{R}}\int_{\mathbb{R}} \omega_1(s)\omega_1^*(v)\omega_2^*(s)\omega_2(v)dsdv\\
&= \frac{B}{2\pi k}\left(\int_{\mathbb{R}} \omega_1(s)\omega_2^*(s)dx\right)\left(\int_{\mathbb{R}} \omega_1^*(v)\omega_2(v)dv\right)\\
&= \frac{B}{2\pi k}|\langle\omega_1(t), \omega_2(t)\rangle|^2.
\end{aligned}$$

Thus completes the proof. □

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4. Applications of the scaled AFQ

In engineering the most important research topics is the detection of LFM signals as they are widely used in communications, information and optical systems. In this section our main goal is to use scaled AFQ in detection of one-component and bi-component LFM signals, respectively.

- **One component LFM signal:** A one-component LFM signal is chosen as

$$\omega(t) = e^{i\left(\vartheta_1 t + \vartheta_2 t^2\right)} \tag{20}$$

where ϑ_1 and ϑ_2 represent the initial frequency and frequency rate of $\omega(t)$, respectively. Then, we obtain the SAFQ of a signal $\omega(t)$ as shown in the following theorem.

Theorem 4.1 *The SAFQ of* $\omega(t) = e^{i\left(\vartheta_1 t + \vartheta_2 t^2\right)}$ *can be presented as*

$$SAF^{\Omega}_{\omega(t)}(\tau, u) = e^{i[k(\vartheta_1 + D)\tau + Eu]}\delta[2k(\vartheta_2 + A)\tau + Bu]. \tag{21}$$

Proof. By Definition 3.1, we have

$$\begin{aligned}
&SAF^{\Omega}_{\omega(t)}(\tau, u)\\
&= \frac{B}{2\pi}\int_{\mathbb{R}} \omega\left(t + k\frac{\tau}{2}\right)\omega^*\left(t - k\frac{\tau}{2}\right)e^{i[(2Ak\tau + Bu)t + Dk\tau + Eu]}dt\\
&= \frac{B}{2\pi}\int_{\mathbb{R}} e^{i\left[\vartheta_1\left(t + k\frac{\tau}{2}\right) + \vartheta_2\left(t + k\frac{\tau}{2}\right)^2\right]}e^{-i\left[\vartheta_1\left(t - k\frac{\tau}{2}\right) + \vartheta_2\left(t - k\frac{\tau}{2}\right)^2\right]}\\
&\qquad\qquad \times e^{i[(2Ak\tau + Bu)t + Dk\tau + Eu]}dt\\
&= \frac{B}{2\pi}\int_{\mathbb{R}} e^{i\left[\vartheta_1 t + \vartheta_1 k\frac{\tau}{2} + \vartheta_2 t^2 + \vartheta_2 tk\tau + \vartheta_2 k^2\frac{\tau^2}{4}\right]}e^{-i\left[\vartheta_1 t - \vartheta_1 k\frac{\tau}{2} + \vartheta_2 t^2 - \vartheta_2 tk\tau + \vartheta_2 k^2\frac{\tau^2}{4}\right]}\\
&\qquad\qquad \times e^{i[(2Ak\tau + Bu)t + Dk\tau + Eu]}dt\\
&= \frac{B}{2\pi}\int_{\mathbb{R}} e^{i[\vartheta_1 k\tau + 2\vartheta_2 tk\tau + (2Ak\tau + Bu)t + Dk\tau + Eu]}dt\\
&= \frac{B}{2\pi}e^{i[k(\vartheta_1 + D)\tau + Eu]}\int_{\mathbb{R}} e^{i[2k(\vartheta_2 + A)\tau + Bu]t}dt\\
&= e^{i[k(\vartheta_1 + D)\tau + Eu]}\delta[2k(\vartheta_2 + A)\tau + Bu],
\end{aligned} \tag{22}$$

□

From above Theorem, we can conclude that the that the SAFQ of a one-component signal (20) are able to generate impulses in (τ, u) plane at a straight line $(Bu + 2k(\vartheta_2 + A)\tau) = 0$ and is dependent on the scaling factor k and the parameter $\Omega = (A, B, C, D, E)$. Therefore, the SAFQ can be applied to the detection of one-component LFM signals and is very useful and effective as there is choice of selecting the scaling factor k and the parameter Ω.

- **Bi-component LFM signal:** Consider the following bi-component LFM signal $\omega(t)$ it is well known that the bi-component LFM signal can be expressed by the summation of two single component LFM signals, i.e.,

$$\omega(t) = \omega_1(t) + \omega_2(t), \tag{23}$$

where $\omega_1(t) = e^{i\left(\xi_1 t + \eta_1 t^2\right)}(\eta_1 \neq 0)$, $\omega_2(t) = e^{i\left(\xi_2 t + \eta_2 t^2\right)}(\eta_2 \neq 0)$ and $\eta_1 \neq \eta_2$. Now using the non-linearity property (8), the SAFQ of the signal $\omega(t)$ given in (23) can be computed as follows:

$$\begin{aligned}
SAF^{\Omega}_{\omega(t)}(\tau, u) &= SAF^{\Omega}_{\omega_1(t)+\omega_2(t)}(\tau, u)\\
&= SAF^{\Omega}_{\omega_1(t)}(\tau, u) + SAF^{\Omega}_{\omega_2(t)}(\tau, u) + SAF^{\Omega}_{\omega_1(t),\omega_2(t)}(\tau, u) + SAF^{\Omega}_{\omega_2(t),\omega_1(t)}(\tau, u)\\
&= e^{i[k(\xi_1+D)\tau+Eu]}\delta[2k(\eta_1 + A)\tau + Bu]\\
&\quad + e^{i[k(\xi_1+D)\tau+Eu]}\delta[2k(\eta_2 + A)\tau + Bu] + SAF^{\Omega}_{\omega_1(t),\omega_2(t)}(\tau, u) + SAF^{\Omega}_{\omega_2(t),\omega_1(t)}(\tau, u).
\end{aligned}$$

The first two terms in last equation stands for the auto-terms of one-component signals, whereas the rest represent the cross terms that are given by

$$\begin{aligned}
&SAF^{\Omega}_{\omega_1(t),\omega_2(t)}(t, u)\\
&= \frac{B}{2\pi}\int_{\mathbb{R}} \omega_1\left(t + k\frac{\tau}{2}\right)\omega_2^*\left(t - k\frac{\tau}{2}\right)e^{i[(2Ak\tau+Bu)t+Dk\tau+Eu]}dt\\
&= \frac{B}{2\pi}\int_{\mathbb{R}} e^{i\left[\xi_1\left(t+k\frac{\tau}{2}\right)+\eta_1\left(t+k\frac{\tau}{2}\right)^2\right]}e^{-i\left[\xi_2\left(t-k\frac{\tau}{2}\right)+\eta_2\left(t-k\frac{\tau}{2}\right)^2\right]}e^{i[(2Ak\tau+Bu)t+Dk\tau+Eu]}dt\\
&= \frac{B}{2\pi}\int_{\mathbb{R}} e^{i\left[\xi_1 t+\xi_1 k\frac{\tau}{2}+\eta_1 t^2+\eta_1 k^2\frac{\tau^2}{4}+\eta_1 tk\tau\right]}e^{-i\left[\xi_2 t-\xi_2 k\frac{\tau}{2}+\eta_2 t^2+\eta_2 k^2\frac{\tau^2}{4}-\eta_2 tk\tau\right]}\\
&\qquad\qquad \times e^{i[(2Ak\tau+Bu)t+Dk\tau+Eu]}dt\\
&= \frac{B}{2\pi}e^{i\left[\frac{\eta_1-\eta_2}{4}k^2\tau^2+\frac{\xi_1+\xi_2+2D}{2}k\tau+Eu\right]}\int_{\mathbb{R}} e^{i(\eta_1-\eta_2)t^2}e^{i[Bu+k(\eta_1+\eta_2+2A)\tau+(\xi_1-\xi_2)]t}dt\\
&= \frac{B}{k}\frac{1}{\sqrt{\pi(\eta_1-\eta_2)}}e^{i\left[\frac{\eta_1-\eta_2}{4}k^2\tau^2+\frac{\xi_1+\xi_2+2D}{2}k\tau+Eu\right]}e^{-i\frac{\left[Bu+k\left(\eta_1+\eta_2+2A\right)t-\left(\xi_1-\xi_2\right)\right]^2}{4\left(\eta_1-\eta_2\right)}},
\end{aligned}$$

similarly

$$\begin{aligned}
&SAF^{\Omega}_{\omega_2(t),\omega_1(t)}(t, u)\\
&= \frac{1}{kb}\frac{1}{\sqrt{\pi(\eta_2-\eta_1)}}e^{i\left[\frac{\eta_2-\eta_1}{4}k^2\tau^2+\frac{\xi_1+\xi_2+2d}{2}k\tau-Eu\right]}e^{-i\frac{\left[Bu+k\left(\eta_2+\eta_1+2A\right)t-\left(\xi_2-\xi_1\right)\right]^2}{4\left(\eta_2-\eta_1\right)}}.
\end{aligned}$$

Hence the SAFQ of a bi-component signal $\omega(t) = \omega_1(t) + \omega_2(t)$ is given by

$$\begin{aligned}
SAF^{\Omega}_{\omega(t)}(\tau, u) &= SAF^{\Omega}_{\omega_1(t)+\omega_2(t)}(\tau, u)\\
&= e^{i[k(\xi_1+D)\tau+Eu]}\delta[2k(\eta_1 + A)\tau + Bu]\\
&\quad + e^{i[k(\xi_1+D)\tau+Eu]}\delta[2k(\eta_2 + A)\tau + Bu]\\
&\quad + \frac{B}{k}\frac{1}{\sqrt{\pi(\eta_1-\eta_2)}}e^{i\left[\frac{\eta_1-\eta_2}{4}k^2\tau^2+\frac{\xi_1+\xi_2+2D}{2}k\tau+Eu\right]}e^{-i\frac{\left[Bu+k\left(\eta_1+\eta_2+2A\right)t-\left(\xi_1-\xi_2\right)\right]^2}{4\left(\eta_1-\eta_2\right)}}\\
&\quad + \frac{1}{kb}\frac{1}{\sqrt{\pi(\eta_2-\eta_1)}}e^{i\left[\frac{\eta_2-\eta_1}{4}k^2\tau^2+\frac{\xi_1+\xi_2+2d}{2}k\tau-Eu\right]}e^{-i\frac{\left[Bu+k\left(\eta_2+\eta_1+2A\right)t-\left(\xi_2-\xi_1\right)\right]^2}{4\left(\eta_2-\eta_1\right)}}.
\end{aligned} \tag{24}$$

It is clear from (24) a that the first two auto-terms are able to generate impulses which the cross terms cannot generate, and therefore, although the existence of cross terms has a certain influence on the detection performance, but the bi-component LFM signal still can be detected. This indicates that the scaled AFQ is also useful and powerful for detecting bi-component LFM signals. Moreover for an adequate value of k and matrix parameter Ω, the scaled AFQ benefits in cross-term reduction while maintaining a perfect time-frequency resolution with clear auto terms angle resolution.

5. Conclusion

Motivated by degree of freedom corresponding to the choice of a factor k in the fractional instantaneous auto-correlation and the extra degree of freedom present in QPFT, we proposed novel scaled AFQ. First, we studied the fundamental properties of the proposed distributions, including the time marginal, conjugate symmetry, non-linearity, time shift, frequency shift, frequency marginal, scaling, inverse and Moyal formula. Finally to show the of advantage of the theory, we provided the applications of the scaled AFQ in the detection of single-component and bi-component linear- frequency-modulated (LFM) signal.

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Author contributions

Both the authors contributed equally in the paper.

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