Chapter

Slepian Pollak Inequality for Multi-Band Wavelets

Mohammad Younus Younus Bhat, Altaf Ahmad Bhat and Javaid Gani Dar

Abstract

The aim of this article is to study Slepian Pollak Inequality (Bell Labs inequality) for multi-band wavelets (M-band wavelets). A version of the already known energy inequality for the wavelets helps to zoom in onto narrow-band high-frequency components of a signal.

Keywords: wavelet, M-band, Slepian Pollak Inequality, Fourier transform, wavelet frame

1. Introduction

To study the high-frequency signals whose bandwidth is low, the classical orthogonal wavelets are not enough. As a result of this, M-band novel orthonormal wavelets were born [1, 2]. The reason behind the greater factor $M(M > 2)$ is that, instead of the standard decomposition which gives logarithmic frequency resolution, the new wavelet decomposition gives rise to a combination of logarithmic and linear frequency resolution and thus designs a more compatible tiling of the time-frequency plane which is absent in 2-band wavelets. For more difference between the novel multi-band wavelets and classical 2-band wavelets, we refer to [3] and the references therein.

Moving to other side, tight wavelet frames have redundancy which is absent in the orthonormal wavelets. By ignoring orthonormality and keeping redundancy, it becomes easier to construct tight wavelet frames that the classical orthonormal wavelets. This construction becomes easier when we use the unitary extension principle (UEP) as catalyst [4]. The research in the area of frame theory is well developed. To mention a few, we refer to [5–7].

The fast growing and emerging branch of research wavelet analysis has became a bridge to connect mathematicians with other researchers like physicians, chemists, engineers, etc. Hence, it becomes quite interesting to study different aspects like uncertainty principle in the context of wavelet analysis. The uncertainty principle inequalities are the basic blocks of signal analysis. Its classical version was studied by Heisenberg, Pauli, Weyl, and Wiener. Going to the application side it is here worth to mention the Slepian Pollak energy inequality (or Bell Labs inequality) and the D. Gabor on a windowed transform. For more details about these inequalities we refer to [8–11].

From the motivational point of view, we consider P as projection operator on wavelet which is in fact subspace of $L^2(\mathbb{R})$. For any signal $f \in L^2(\mathbb{R})$ with finite energy, we consider a time-truncation operator T, which in mathematical terms is defined by $Tf(t) = f(t)\chi_{[a,b]}(t)$, with usual characteristic function $\chi_{[a,b]}$ on any segment $[a,b]$. Let the signal have unit norm, i.e. $\|f\|_2 = 1$ and $\|Tf\|_2$ has certain value in $[0,1]$. Then the authors in [11] considered the possible values of $\|Pf\|_2$, which is in fact a version of uncertainty principle as there is uncertainty between time-truncation operator T and projection operator P. To be precise they solved Fourier projection transform and expressed it in terms of well-known Bell Labs inequality.

Motivated by the work of [11], we consider its generalisation on M-band wavelets. The rest of paper is organised as follows. The paper is organised in the following manner. In Section 2, we take some review of multi-band wavelets. In Section 3, we establish the Slepian Pollak energy inequality.

2. Preliminaries on multi-band wavelet frames

Let us begin with some preliminaries on M-band wavelet frames. We make use of the symbols $\mathbb{N}, \mathbb{N}_0, \mathbb{Z}$, and $\mathbb{R}$ for the sets of all natural numbers, nonnegative integers, integers, and real numbers, respectively. Whereas the identity matrix of size $M > 2$ is denoted by I_M.

For any function $f \in L^1(\mathbb{R})$, we define its Fourier transform as:

$$\hat{f}(\zeta) = \int_{\mathbb{R}} f(x) e^{-i\zeta x} dx, \quad \zeta \in \mathbb{R}$$

and its inverse is

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\zeta) e^{i\zeta x} d\zeta, \quad x \in \mathbb{R}.$$

For given $\mathfrak{G} := \{\mathfrak{g}_1, \dots, \mathfrak{g}_L\} \subset L^2(\mathbb{R})$, define the M-band wavelet system

$$X(\mathfrak{G}) := \left\{ \mathfrak{g}_{j,k}^{\ell} : 1 \le \ell \le L; j, k \in \mathbb{Z} \right\} \tag{1}$$

where $\mathfrak{g}_{j,k}^{\ell} = M^{j/2} \mathfrak{g}^{\ell}\left(M^j . - k\right)$. The wavelet system $X(\mathfrak{G})$ is treated as the *M-band wavelet frame*, or simply a *wavelet frame*, whenever we have positive numbers $0 < \mathfrak{A} \le \mathfrak{B} < \infty$ such that

$$\mathfrak{A}\|f\|_2^2 \le \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \left| \left\langle f, \mathfrak{g}_{j,k}^{\ell} \right\rangle \right|^2 \le \mathfrak{B}\|f\|_2^2. \tag{2}$$

holds for all $f \in L^2(\mathbb{R})$.

The constant $\mathfrak{A}$ which is in is called fact largest the *lower* frame bound, whereas the smallest constant $\mathfrak{B}$ is treated as the *upper wavelet frame bounds*. We have a *tight wavelet frame* whenever $\mathfrak{A} = \mathfrak{B}$. Moreover, the generated system will be a *Parseval wavelet frame* if $\mathfrak{A} = \mathfrak{B} = 1$. Here, every function $f \in L^2(\mathbb{R})$ is reshaped as

$$f(x) = \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \left\langle f, \mathfrak{g}_{j,k}^{\ell} \right\rangle \mathfrak{g}_{j,k}^{\ell}(x). \tag{3}$$

Framelet systems are often constructed from the construction of MRA, which is in turn depends on refinable functions. We say function $\mathfrak{f} \in L^2(\mathbb{R})$ an M-refinable whenever it satisfies a refinement equation:

$$\mathfrak{f}(x) = \sum_{k\in\mathbb{Z}} h_0[k]\,\mathfrak{f}(Mx-k). \tag{4}$$

for some $h_0 \in l^2(\mathbb{Z})$. Its Fourier transform is

$$\hat{\mathfrak{f}}(\zeta) = m_0\left(\frac{\zeta}{M}\right)\hat{\mathfrak{f}}\left(\frac{\zeta}{M}\right), \tag{5}$$

where

$$m_0(\zeta) = \frac{1}{M}\sum_{k\in\mathbb{Z}} h_0[k]e^{ik\zeta},$$

is a 2π-periodic measurable function in $L^\infty[-\pi,\pi]$ and is often called the *refinement symbol* of $\mathfrak{f}$. We further assume that:

$$\lim_{\zeta\to 0}\hat{\mathfrak{f}}(\zeta) = 1, \quad \text{and} \quad \sum_{k\in\mathbb{Z}}\left|\hat{\mathfrak{f}}(\zeta+2k\pi)\right|^2 \in L^\infty[-\pi,\pi]. \tag{6}$$

With $\mathfrak{f} \in L^2(\mathbb{R})$ a refinable and compactly supported function, consider the space V_0, which is created by $\{\mathfrak{f}(\cdot-k) : k\in\mathbb{Z}\}$ and $V_j = \left\{\mathfrak{f}\left(M^j\cdot\right) : \mathfrak{f}\in V_0\right\}$, $j\in\mathbb{Z}$. Obviously if $\mathfrak{f}$ is compactly supported, the $\left\{V_j\right\}_{j\in\mathbb{Z}}$ ends with an MRA. Note that MRA is a collection of closed subspaces $\left\{V_j\right\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ which obeys: (i) $V_j \subset V_{j+1}, j\in\mathbb{Z}$, (ii) $\cup_{j\in\mathbb{Z}}V_j$ is dense in $L^2(\mathbb{R})$, and (iii) $\cap_{j\in\mathbb{Z}}V_j = \{0\}$. Let $\mathfrak{G} := \{\mathfrak{g}_1, \ldots, \mathfrak{g}_L\} \subset V_1$, then

$$\hat{\mathfrak{g}}_\ell(\zeta) = m_\ell\left(\frac{\zeta}{M}\right)\hat{\mathfrak{f}}\left(\frac{\zeta}{M}\right), \tag{7}$$

where

$$m_\ell(\zeta) = \frac{1}{M}\sum_{k\in\mathbb{Z}} h_\ell[k]e^{ik\zeta}, \quad \ell = 1, \ldots, L$$

are the 2π-periodic measurable functions in $L^\infty[-\pi,\pi]$ which are named as *framelet symbols*. The well-known catalyst *unitary extension principle* (UEP) incorporates a sufficient condition on the system $\mathfrak{G}$ so that the resulting M-band system $X(\mathfrak{G})$ becomes a tight frame of $L^2(\mathbb{R})$. It were Han and Cheng [6] who provided a complete characterisation of these multi-band tight wavelet frames using the machinery of the unitary extension principle. In fact they used the following tool.

Theorem 2.1. [6] Suppose that the refinable function $\mathfrak{f}$ and the framelet symbols $m_0, m_1, \dots, m_L$ satisfy (4)–(7). Define $\mathfrak{g}_1, \dots, \mathfrak{g}_L$ by (7). Let $\mathscr{M}(\zeta) = \left\{ m_\ell\left(\zeta + \frac{2\pi p}{M}\right)\right\}_{\ell,p=0}^{M-1}$ such that $\mathscr{M}(\zeta)\mathscr{M}^*(\zeta) = I_M$, for $a.e$ $\zeta \in \sigma(V_0) := \left\{\zeta \in [-\pi,\pi] : \sum_{k\in\mathbb{Z}}\left|\hat{\mathfrak{f}}(\zeta + 2k\pi)\right|^2 \neq 0\right\}$, then M-band wavelet system $X(\mathfrak{g})$ forms a tight wavelet frame for $L^2(\mathbb{R})$ with frame bound 1.

3. Projection on MRA spaces

Define the projection $P : f \to f_m \in V_m$ by

$$Pf(x) = f_m(x) = \sum_{\ell=1}^{L}\sum_{j=-\infty}^{m-1}\sum_{k=-\infty}^{\infty}\left\langle f, \mathfrak{g}_{j,k}^{\ell}\right\rangle \mathfrak{g}_{j,k}^{\ell}(x) = \sum_{k=-\infty}^{\infty}\left\langle f, \mathfrak{f}_{j,k}\right\rangle \mathfrak{f}_{j,k}(x). \tag{8}$$

Lemma 3.1. *Let $\mathfrak{f}$ be a continuous scaling function of bounded variation on $\mathbb{R}$. Suppose one of the following equations hold*

$$\sum_{k\in\mathbb{Z}}\left|\hat{\mathfrak{f}}(\zeta + 2\pi k)\right| < A < \infty, \quad \text{for a.e.}\,\zeta \in \mathbb{R}, \tag{9}$$

$$\sum_{k\in\mathbb{Z}}|\mathfrak{f}(x-k)| < B < \infty, \quad \text{for a.e.}\,x \in \mathbb{R}. \tag{10}$$

Then for any integer j and $f \in L^2(\mathbb{R})$, we have

$$f_j(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\zeta)\hat{\mathfrak{f}}(M^{-j}\zeta)e^{i\zeta x}\sum_{n=-\infty}^{\infty}\hat{\mathfrak{f}}(M^{-j}\zeta + 2\pi n)e^{i2\pi n M^j x}d\zeta. \tag{11}$$

Proof. Using Parseval's identity, we have from Eq. (8)

$$\begin{aligned}
f_j(x) &= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{f}(\zeta)\overline{\hat{\mathfrak{f}}\left(M^{-j}\zeta\right)}e^{ikM^{-j}\zeta}\mathfrak{f}\left(M^j x - k\right)d\zeta \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\zeta)\overline{\hat{\mathfrak{f}}\left(M^{-j}\zeta\right)}\sum_{k=-\infty}^{\infty}\left\{e^{ikM^{-j}\zeta}\mathfrak{f}\left(M^j x - k\right)\right\}d\zeta \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\zeta)\overline{\hat{\mathfrak{f}}\left(M^{-j}\zeta\right)}e^{i\zeta x}\sum_{k=-\infty}^{\infty}\left\{e^{-i\left(M^j x - k\right)M^{-j}\zeta}\mathfrak{f}\left(M^j x - k\right)\right\}d\zeta \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\zeta)\overline{\hat{\mathfrak{f}}\left(M^{-j}\zeta\right)}e^{i\zeta x}\sum_{n=-\infty}^{\infty}\left\{e^{i2\pi n M^j x}\hat{\mathfrak{f}}\left(M^{-j}\zeta + 2\pi n\right)\right\}d\zeta,
\end{aligned}$$

where we have used Poisson Summation formula, because of Eq. (10). This completes the proof. □

Lemma 3.2. *Let $\mathfrak{f}$ be as in Lemma 3.1 and P be the projection operator. If $\mathcal{F}$ is a bounded family of square integrable functions, then its image $P(\mathcal{F})$ is uniformly bounded family on $\mathbb{R}$.*

Proof. Let $\mathfrak{f}$ satisfy condition (9). Then from (11), we have by Schawrtz Inequality

$$
\begin{aligned}
\left|f_j(x)\right| &\leq \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\hat{f}(\zeta)\right|\left|\hat{\mathfrak{f}}\left(M^{-j}\zeta\right)\right|\left|\sum_{n=-\infty}^{\infty}\hat{\mathfrak{f}}\left(M^{-j}\zeta+2\pi n\right)e^{i\left(\zeta+2\pi nM^j\right)x}\right|d\zeta \\
&\leq \frac{1}{2\pi}\left\|\hat{f}\right\|_2\left\{\int_{-\infty}^{\infty}\left|\hat{\mathfrak{f}}\left(M^{-j}\zeta\right)\right|^2\left(\sum_{n=-\infty}^{\infty}\left|\hat{\mathfrak{f}}\left(M^{-j}\zeta+2\pi n\right)\right|\right)^2 d\zeta\right\}^{1/2} \\
&\leq \frac{1}{\sqrt{2\pi}}A\|f\|_2\left(\int_{-\infty}^{\infty}\left|\hat{\mathfrak{f}}\left(M^{-j}\zeta\right)\right|^2 d\zeta\right)^{1/2} \\
&\leq M^{j/2}A\|f\|_2.
\end{aligned}
$$

Moreover, if $\mathfrak{f}$ satisfies Eq. (10), then we have

$$
\left|f_j(x)\right| \leq \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\hat{f}(\zeta)\right|\left|\hat{\mathfrak{f}}\left(M^{-j}\zeta\right)\right|\left|\sum_{n=-\infty}^{\infty}\mathfrak{f}\left(M^j x-k\right)\right|d\zeta \leq M^{j/2}B\|f\|_2.
$$

Theorem 3.3. *Let $\mathfrak{f}$ satisfy the conditions of Lemma 3.1, P be the projection operator, and T be the truncation operator. Then PT is a compact operator.*

Proof. Define

$$
\Delta_f(\epsilon) = \sup_x\left|f(x+\epsilon)-f(x)\right|.
$$

where $\epsilon > 0$. Let $\mathfrak{f}$ satisfy the condition (9), then

$$
\begin{aligned}
\Delta_{f_j}(\delta) &\leq \int_{-\infty}^{\infty}\left|\hat{f}(\zeta)\right|\left|\hat{\mathfrak{f}}\left(M^{-j}\zeta\right)\right|\sum_{n=-\infty}^{\infty}\left|\hat{\mathfrak{f}}\left(M^{-j}\zeta+2\pi n\right)\right|\left|e^{i\left(\zeta+2\pi nM^j\right)\epsilon}-1\right|d\zeta \\
&\leq \frac{\|f\|_2}{\sqrt{2\pi}}\left\{\int_{-\infty}^{\infty}\left(\left|\hat{\mathfrak{f}}\left(M^{-j}\zeta\right)\right|\sum_{n=-\infty}^{\infty}\left|\hat{\mathfrak{f}}\left(M^{-j}\zeta+2\pi n\right)\right|\left|e^{i\left(\zeta+2\pi nM^j\right)\epsilon}-1\right|\right)^2 d\zeta\right\}^{1/2}.
\end{aligned}
$$

Since

$$
\left|\hat{\mathfrak{f}}\left(M^{-j}\zeta\right)\right|^2\left(\sum_{n=-\infty}^{\infty}\left|\hat{\mathfrak{f}}\left(M^{-j}\zeta+2\pi n\right)\right|\left|e^{i\left(\zeta+2\pi nM^j\right)\epsilon}-1\right|\right)^2 \leq 4A^2\left|\hat{\mathfrak{f}}\left(M^{-j}\zeta\right)\right|^2 \in L^1(\mathbb{R})
$$

and by (9), the series

$$
\sum_{n=-\infty}^{\infty}\left|\hat{\mathfrak{f}}\left(M^{-j}\zeta+2\pi n\right)\right|\left|e^{i\left(\zeta+2\pi nM^j\right)\delta}-1\right| \to 0 \quad \text{as} \quad \epsilon \to 0,
$$

Lebesque Dominated convergence theorem implies that

$$
\lim_{\epsilon\to 0}\Delta_{f_j}(\epsilon) = 0.
$$

By taking inverse Fourier transform to the above, the result also holds. Thus for any bounded family $\mathcal{F}$ in $L^2(\mathbb{R})$, its image $P(\mathcal{F})$ is equicontinuous and uniformly bounded family on $\mathbb{R}$. Hence, according to Ascoli's Theorem, PT is a compact operator.

Thus with the above results in hand we are in a position to say that we have proved the $M-$ band version of Bell Labs inequality. To be precise, mathematically we have proved the following result:

Let $\mathfrak{f}$ *be as in the Lemma 2, P be the projection operator, and T be the truncation operator, then* (P, T) *is a pair satisfying Bell Labs inequality*

$$\arccos \frac{\|Tf\|_2}{\|f\|_2} + \arccos \frac{\|Pf\|_2}{\|f\|_2} \geq \arccos\|Tf\|_2.$$

Acknowledgements

We are deeply indebted to the referee for his/her valuable suggestions which greatly improved the presentation of this paper.

Conflict of interest

The authors declare no conflict of interest.

Data availability

Data will be available on the request to authors.

Author details

Mohammad Younus Younus Bhat[1*], Altaf Ahmad Bhat[2] and Javaid Gani Dar[3]

1 Department of Mathematical Sciences, Islamic University of Science and Technology, Kashmir, India

2 Sciences and Mathematics Unit, University of Technology and Applied Sciences, Salalah, Sultanate of Oman

3 Department of Applied Sciences, Symbiosis Institute of Technology, Symbiosis International (Deemed University), Pune, India

*Address all correspondence to: gyounusg@gmail.com

References

[1] Shah FA, Bhat MY. On framelet kernels of M-band wavelet frames. Gulf Journal of Mathematics. 2015;**3**(4):59-66

[2] Steffen P, Heller PN, Gopinath RA, Burrus CS. Theory of regular M-band wavelet bases. IEEE Transactions on Signal Processing. 1993;**41**(12): 3497-3510

[3] Bi N, Dai X, Sun Q. Construction of compactly supported M-band wavelets. Applied and Computational Harmonic Analysis. 1999;**6**:113-131

[4] Ron A, Shen Z. Affine systems in $L^2(R^d)$: The analysis of the analysis operator. Journal of Functional Analysis. 1997;**148**:408-447

[5] Chai A, Shen Z. Deconvolution: A wavelet frame approach. Numerische Mathematik. 2007;**106**:529-587

[6] Han JC, Cheng Z. The construction of M-band tight wavelet frames. In: Proceedings of the Third Intemational Conference on Machine Learning and Cybemetics; Shanghai. 2004. pp. 26-29

[7] Rakotomamonjy A, Canu S. Frames, reproducing kernels, regularization and learning. Journal of Machine Learning Research. 2005;**6**:1485-1515

[8] Benedetto JJ. Uncertainty principle inequalities and spectrum estimation. In: Byrnes JS, Byrnes JL, editors. Recent Advances in Fourier Analysis and Its Applications. Dordrecht: Iiluwer Acad. Publ; 1990. pp. 143-182

[9] Folland GB, Sitaram A. The uncertainty principles: A mathematical survey. Journal of Fourier Analysis and Applications. 1997;**106**:209-238

[10] Havin V, Joricke B. The Uncertainty Principle in Harmonic Analysis, A Series of Modern Survey in Mathematics No 28. Berlin: Springer-Verlag; 1994

[11] Karanikas K, Proios G. The slepian-pollak energy inequality for wavelets and frames. Applicable Analysis. 1991;**70**: 321-329